Well-foundedness is sufficient for completeness of Ordered Paramodulation

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Abstract. For many years all known completeness results for Knuth-Bendix completion and ordered paramodulation required the term ordering $\succ$ to be well-founded, monotonic and total(izable) on ground terms. Then, it was shown that well-foundedness and the subterm property were enough for ensuring completeness of ordered paramodulation.
Here we show that the subterm property is not necessary either. By using a new restricted form of rewriting we obtain a completeness proof of ordered paramodulation for Horn clauses with equality where well-foundedness of the ordering suffices. Apart from the fundamental interest of this result, some potential applications motivating the interest of dropping the subterm property are given.

1 Introduction

Knuth-Bendix-like completion techniques and their extensions to ordered paramodulation for first-order clauses are among the most successfull methods for automated deduction with equality [BG98,NR01]. For many years all known completeness results for Knuth-Bendix completion and ordered paramodulation required the term ordering $\succ$ to be well-founded, monotonic and total (or extendable to a total ordering) on ground terms [HR91,BDH86,BD94,BG94], until in [BGNR99], the monotonicity requirement was dropped and well-foundedness and the subterm property were shown to be sufficient (note that any such ordering can be totalized without loosing these two properties).

After this, the fundamental question arises whether more requirements can be dropped. In this paper we prove that for ordered paramodulation on Horn clauses with equality the subterm property can be dropped as well. Therefore only well-foundedness is required.

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These results are a new important step in the theory of paramodulation, which somehow shows the power of ordered paramodulation regardless of the properties of the ordering which is used and leaves as last question whether even well-foundedness is not necessary.

Apart from its theoretical value, there are at least two other potential applications of these results. The first one is in deduction modulo built-in equational theories E, since for many E, finding *E-compatible* orderings fulfilling the required properties is extremely complex (for example for the associative and commutative (AC) case) or even impossible (for instance in presence of idempotence axioms [BGNR99]).

The second application comes from the fact that, sometimes, a goal-oriented (ordered) paramodulation proof can only be obtained if the ordering contradicts the subterm property. Let us show an example.

Example 1. Consider the following set of equations:

$$\begin{aligned} y^{log_y x} &\simeq x \\ log_y y^x &\simeq x \\ x^{y+z} &\simeq x^y \cdot x^z \end{aligned}$$

and the goal $\qquad log_a b + log_a c \simeq log_a(b \cdot c).$

With the following well-founded ordering

$$\begin{array}{rll} y^{log_y x} &\succ x & \\ x &\succ log_y y^x & \text{if } x \text{ is not headed by } log \\ log_y y^x &\succ x & \text{if } x \text{ is headed by } log \\ x^{y+z} &\succ x^y \cdot x^z & \end{array}$$

which contradicts the subterm property, we have a goal-oriented proof by ordered paramodulation (note that some variable renamings have been applied).

$$\dfrac{\dfrac{\boxed{y_1^{log_{y_1} x_1}} \simeq x_1 \qquad \dfrac{\boxed{y_1^{log_{y_1} x_1}} \simeq x_1 \qquad \dfrac{\boxed{x_1^{y_1+z_1}} \simeq x_1^{y_1} \cdot x_1^{z_1} \qquad \dfrac{\boxed{x} \simeq log_y y^x \qquad \boxed{log_a b + log_a c} \not\simeq log_a(b \cdot c)}{log_y \boxed{y^{log_a b + log_a c}} \not\simeq log_a(b \cdot c)}}{log_y(\boxed{y^{log_a b}} \cdot y^{log_a c}) \not\simeq log_a(b \cdot c)}}{log_a(b \cdot \boxed{a^{log_a c}}) \not\simeq log_a(b \cdot c)}}{log_a(b \cdot c) \not\simeq log_a(b \cdot c)}}{\square}$$

Note that there is no such a fully goal-oriented proof with respect to a well-founded ordering including the subterm relation, because no paramodulation step could be applied on the goal: for the first two equations, due to the subterm property, only the term on the left can be used and it does not unify with any subterm of the goal; for the last equation none of both sides overlaps with the goal. $\square$

In this paper we introduce techniques for dropping the subterm (as well as the monotonicity) requirement. Our technique (given in Sections 4 and 5) is a new variant of the model generation technique, where, surprisingly, the ground rewrite system R that defines the model may not be terminating. For this reason, we define a *new* restricted form of rewriting, called *blocked rewriting* ($\mathcal{B}$-rewriting for short), which is terminating by definition.

$\mathcal{B}$-rewriting works on *abstracted terms*, denoted by $t \cdot \gamma$, which are terms split into two parts: the *abstraction (or term) part* t and the *substitution part* γ, such that $t\gamma$ is a term (this formalism corresponds to the *closures* defined for clauses in [BGLS95]). No rewrite steps can be applied to the γ part and when we rewrite $t \cdot \gamma$ with a rule $l \to r$ at a non-variable position p in t we obtain $t[x]_p \cdot (\gamma \cup \{x \mapsto r\})$ for some fresh variable x. It is obvious that this rewrite relation terminates since each step strictly reduces the size of the abstraction part t.

Example 2. Consider the set of equations E containing only

$$g(a) \simeq a$$

and an ordering such that $a \succ g(a)$ and $g^k(a) \succ g^j(a)$ with $k \geq 2$ and $k > j \geq 0$. In this ordering the minimal term is $g(a)$.

If we close this set E with respect to the ordered paramodulation rule, we obtain a set of equations of the form

$$g^k(a) \simeq g^j(a) \text{ with } k > j \geq 0.$$

which are oriented as

$$\begin{aligned} a &\to g(a) \\ g^k(a) &\to g^j(a) \quad \text{for all } k \geq 2 \text{ and } k > j \geq 0 \end{aligned}$$

Then the following set R containing the two rules

$$\begin{aligned} a &\to g(a) \\ g(g(a)) &\to g(a) \end{aligned}$$

is a confluent (although overlapping) rewrite system for $\mathcal{B}$-rewriting, and furthermore all equations in the closure can be proved by $\mathcal{B}$-rewriting using R. In particular, all $g^k(a) \simeq g^j(a)$ can be proved by joining both sides into the same abstracted term $x \cdot \{x \mapsto g(a)\}$. Therefore, for any consequence of E there is a $\mathcal{B}$-rewrite proof using R. $\square$

In our completeness proof a (possibly infinite) rewrite system like the R above is generated. Proving confluence of R is essential but, since the system may be overlapping, this property does not follow easily. Once we have proved it, we can show that there is a $\mathcal{B}$-rewrite proof with R for any consequence of the unit positive equations. Finally, this is used to prove the completeness of ordered paramodulation for first-order Horn clauses with eager *selection* of negative literals.

Besides the proof of confluence (which is the hardest and more technical part), the core of the proof is rather simple and similar to the one in [BGNR99], since it still uses induction on the $\mathcal{B}$-rewrite relation with R (which is well-founded) for the completeness result.

The paper is structured as follows. The next section provides the basic definitions. In Section 3 $\mathcal{B}$-rewriting is introduced and its main properties stated. Section 4 is devoted to the equational case. There, the original part of our proof technique is presented and the difficulties of proving the confluence of the generated rewrite system that defines the model are outlined. In section 5, completeness for Horn clauses is proved using the results obtained for the equational case. Finally in section 6 conclusions and future developments are discussed. All proofs can be found in [BR02].

2 Basic notions

We use the standard definitions of [DJ90]. $T(\mathcal{F},\mathcal{X})$ $(T(\mathcal{F}))$ is the set of (ground) terms over a set of symbols $\mathcal{F}$ and a denumerable set of variables $\mathcal{X}$ (over $\mathcal{F}$). By $vars(t)$ we denote the set of variables occurring in a term t. The size of a term t is denoted by $|t|$ and its size without counting the variables by $|t|_v$.

The subterm of t at *position* p is denoted by $t|_p$, the result of replacing $t|_p$ by s in t is denoted $t[s]_p$, and syntactic equality of terms is denoted by $\equiv$. Let t be a term in $T(\mathcal{F},\mathcal{X})$ and M a set of terms in $T(\mathcal{F},\mathcal{X})$. We denote by $P_M(t)$ the set of positions p of t s.t. there is some prefix p' of p s.t. $t|_{p'} \in M$, i.e. if $t|_p \in M$ then $P_M(t)$ includes p and all positions below p. By $P_{\mathcal{X}}(t)$ we denote the set of variable positions of t, i.e. $P_{vars(t)}(t)$. Let σ be a ground substitution. By $P(t,\sigma)$ we denote the set of positions of $t\sigma$ which are in $P_{\mathcal{X}}(t)$ or below some position in $P_{\mathcal{X}}(t)$.

If $\rightarrow$ is a binary relation, then $\leftarrow$ is its inverse, $\leftrightarrow$ is its symmetric closure, $\xrightarrow{+}$ is its transitive closure and $\xrightarrow{*}$ is its reflexive-transitive closure. We write $s \xrightarrow{!} t$ if $s \xrightarrow{*} t$ and there is no t' such that $t \rightarrow t'$. Then t is called *irreducible* and a *normal form* of s (wrt. $\rightarrow$). If a term s has exactly one normal form then it is denoted by $s!_{\rightarrow}$. The relation $\rightarrow$ is *well-founded* or *terminating* if there exists no infinite sequence $s_1 \rightarrow s_2 \rightarrow \ldots$ and it is *confluent* or *Church-Rosser* if the relation $\leftarrow^* \circ \rightarrow^*$ is contained in $\rightarrow^* \circ \leftarrow^*$. It is *locally confluent* if $\leftarrow \circ \rightarrow \ \subseteq \ \rightarrow^* \circ \leftarrow^*$. By Newman's lemma, terminating locally-confluent relations are confluent. A relation $\rightarrow$ on terms is *monotonic* if $s \rightarrow t$ implies $u[s]_p \rightarrow u[t]_p$ for all terms s,

t and u and positions p. A *congruence* is a reflexive, symmetric, transitive and monotonic relation on terms.

An *equation* is a multiset of terms $\{s, t\}$, denoted $s \simeq t$ or, equivalently, $t \simeq s$. By $gnd(s \simeq t)$ we denote the set of ground instances of $s \simeq t$. Let E be a set of equations. By $gnd(E)$ we denote the set of ground instances of all equations in E.

A first-order clause is a pair of finite multisets of equations Γ (the *antecedent*) and Δ (the *succedent*), denoted by $\Gamma \to \Delta$. Equations in Γ are called *negative literals* and equations in Δ are called *positive literals*. A Horn clause is a clause with at most one positive literal. The empty clause $\square$ is a clause where both Γ and Δ are empty.

A *rewrite rule* is an ordered pair of terms (s, t), written $s \to t$, and a set of rewrite rules R is a *term rewrite system* (TRS). By $Left(R)$ and $Right(R)$ we denote, respectively, the set of left-hand sides of rules in R and the set of right-hand sides of rules in R. The rewrite relation with R on $T(\mathcal{F}, \mathcal{X})$, denoted $\to_R$, is the smallest monotonic relation such that $l\sigma \to_R r\sigma$ for all $l \to r \in R$ and all substitution σ, and if $s \to_R t$ then we say that *s rewrites into t* with R. R is called terminating, confluent, etc. if $\to_R$ is.

A (strict partial) *ordering* on $T(\mathcal{F}, \mathcal{X})$ is an irreflexive transitive relation $\succ$. It is stable under substitutions if $s \succ t$ implies $s\sigma \succ t\sigma$ for all substitutions σ. It fulfils the *subterm property* if $\succ \supseteq \rhd$, where $\rhd$ denotes the strict subterm ordering.

Definition 1. *A* west-ordering *is a well-founded ordering on $T(\mathcal{F})$ that fulfils the subterm property and that is total on $T(\mathcal{F})$ (it is called* west *after well-founded, subterm and total).*

Every well-founded ordering can be totalized [Wec91], and hence every well-founded ordering satisfying the subterm property can be extended to a west-ordering.

3 $\mathcal{B}$-rewriting

Now we define a special kind of rewrite relation, called *blocked rewriting* ($\mathcal{B}$-rewriting for short), which is always terminating. This relation restricts standard rewriting by mainly forbidding rewrite steps on right-hand sides introduced by previous rewrite steps. Roughly speaking, after every rewrite step the introduced right-hand side becomes blocked.

In order to express this blocking we use *abstracted terms* which have variables in the blocked positions. The blocked part of the term is stored in a substitution.

Definition 2. *Let R be a TRS. Let γ be a ground substitution and s a linear term with $vars(s) \subseteq Dom(\gamma)$ and $Im(\gamma) \subseteq Right(R)$. Then $s \cdot \gamma$ is called an* abstracted term *wrt. R, and an* abstraction *of $s\gamma$ wrt. R.*

In the following, we write s instead of $s \cdot \gamma$ whenever γ denotes the empty substitution.

Two abstracted terms $s \cdot \gamma$ and $t \cdot \delta$ are equivalent, denoted by $s \cdot \gamma \equiv t \cdot \delta$, if s and t are equal up to renaming of variables and $s\gamma \equiv t\delta$.

An abstracted term $t' \cdot \delta$ is called an argument abstraction *wrt. R of a term $t = f(t_1, \ldots, t_n)$ if $t' = f(t'_1, \ldots, t'_n)$ and any $t'_i \cdot \delta$ is an abstraction of t_i which cannot be further abstracted. We (ambiguously) write $arg\_abs(t, R)$ to denote any argument abstraction of t wrt. R, since all argument abstractions of t wrt. R are equivalent.*

Given an abstracted term $s \cdot \gamma$ wrt. R, the set of *blocked positions* the set of positions q in $s\gamma$ such that q is below or equal to a variable position p in s (i.e. $p \in P_{\mathcal{X}}(s)$).

Now, we can define $\mathcal{B}$-rewriting as a rewrite relation on abstracted terms.

Definition 3. *Let R be a TRS and $s \cdot \gamma$ an abstracted term wrt. R. Then $s \cdot \gamma \curvearrowright_R s[x]_p \cdot (\gamma \cup \{x \mapsto r\})$ if $s|_p \notin \mathcal{X}$, $x \in \mathcal{X} \setminus vars(s)$, and either $s\gamma|_p \to r \in R$ or $s\gamma|_p \equiv r \in Right(R)$.*

If $s\gamma|_p \equiv r \in Right(R)$ we refer to the rewrite step as an abstraction step*.*

We will ambiguosly write $s \cdot \gamma \curvearrowright_S t \cdot \delta$ to denote a non-abstraction step using a rule of a subset S of R.

$\mathcal{B}$-rewriting fulfils the following properties:

Lemma 1. *Let R be a TRS. Then $\overset{+}{\curvearrowright}_R$ is a well-founded ordering.*

Lemma 2. *Let R be a TRS. Then $\overset{+}{\curvearrowright}_R$ is compatible with the equivalence on abstracted terms, i.e., if $s' \cdot \gamma' \equiv s \cdot \gamma \overset{+}{\curvearrowright}_R t \cdot \delta \equiv t' \cdot \delta'$ then $s' \cdot \gamma' \overset{+}{\curvearrowright}_R t' \cdot \delta'$.*

Lemma 3. *Let R be a TRS. Then $\overset{+}{\curvearrowright}_R$ is a monotonic relation, in the sense that $s \cdot \gamma \overset{+}{\curvearrowright}_R t \cdot \delta$ implies $C[s]_p \cdot (\rho \cup \gamma) \overset{+}{\curvearrowright}_R C[t]_p \cdot (\rho \cup \delta)$, where $C \cdot \rho$ is an abstracted context (i.e. an abstracted term with a hole, also denoted by $C[\,]_p \cdot \rho$).*

We must note that what identifies a rewrite step is the rule which is applied and the position where it is applied, regardless of the context. Similarly, a rewrite sequence is identified by a sequence of steps. However, we usually refer (ambiguously) to a rewrite sequence making the context where it is applied explicit. Then, given a TRS R, we refer to $s \cdot \gamma \overset{*}{\curvearrowright}_R t \cdot \delta$ as a rewrite sequence with $\curvearrowright_R$ of $t \cdot \delta$ from $s \cdot \gamma$.

Now we define two restricted versions of $\curvearrowright_R$, denoted by $\curvearrowright_{\underline{R}}$ and $\curvearrowright_{\overline{R}}$, which respectively consist of rewriting (by non-abstraction steps) only over blocked positions and only over non-blocked positions. Recall that the only forbidden steps for $\curvearrowright_R$ are the ones at or under blocked positions. For instance in the abstracted term $f(a, x) \cdot \{x \mapsto g(a)\}$, we can rewrite with $\curvearrowright_{\underline{R}}$ only at position λ and with $\curvearrowright_{\overline{R}}$ only at position 1.

Definition 4. *Let R be a TRS and $s \cdot \gamma$ an abstracted term wrt. R.*

We write $s \cdot \gamma \curvearrowright_{\underline{R}} s[x]_p \cdot (\gamma \cup \{x \mapsto r\})$ if $s \cdot \gamma \curvearrowright_R s[x]_p \cdot (\gamma \cup \{x \mapsto r\})$, $s\gamma|_p \to r \in R$ and $vars(s|_p) \neq \emptyset$.

We write $s \cdot \gamma \curvearrowright_{\overline{R}} s[x]_p \cdot (\gamma \cup \{x \mapsto r\})$ if $s \cdot \gamma \curvearrowright_R s[x]_p \cdot (\gamma \cup \{x \mapsto r\})$, $s\gamma|_p \to r \in R$ and $vars(s|_p) = \emptyset$.

4 Paramodulation with equations

In the following, let $\succ$ be a given well-founded ordering (total) on ground terms.

Definition 5. *The inference rule of (equational)* ordered paramodulation *with respect to* $\succ$ *is:*

$$\frac{l \simeq r \qquad s \simeq t}{(s[r]_p \simeq t)\sigma}$$

where $\sigma = mgu(s|_p, l)$*, the most general unifier of* $s|_p$ *and* l*, where* l *is maximal in its premise (that is, for some ground substitution* θ*, it holds that* $l\sigma\theta \succ r\sigma\theta$*) and* $s|_p$ *is not a variable or, if* $s|_p$ *is a variable* x*, then* $x \in vars(t)$*,* $p = \lambda$ *and* l *is also a variable.*

Note that this last condition implies that in very restrictive cases we have to perform an inference on a variable position. However, these potential inferences on variable positions are not prolific at all, since both s *and* l *should be variables (with* s *occurring in* t*, which may happen since the ordering does not need to satisfy the subterm property). Hence the only possible inference has* $r\sigma \simeq t$ *as conclusion where* σ *is the variable renaming of* l *by* s*.*

We associate a ground TRS $\vec{E}$ to a set of equations E.

Definition 6. *Let* E *be a set of equations. We define* $\vec{E}$ *as the ground TRS obtained by orienting wrt.* $\succ$ *the ground instances of all equations in* E*. That is,*

$$\vec{E} = \{s \to t \mid s \simeq t \in gnd(E) \wedge s \succ t\}.$$

Example 3. Following Example 2 in the introduction, as said, the closure by ordered paramodulation of the equation $a \simeq g(a)$, consists of all equations of the form $g^k(a) \simeq g^j(a)$ with $k > j \geq 0$. Then, with the considered well-founded ordering, $\vec{E}$ consists of the rules

$$\begin{array}{ll} a \to g(a) & \\ g^k(a) \to g^j(a) & \text{for all } k \geq 2 \text{ and } k > j \geq 0 \end{array}$$

□

In the following we assume E to be a set of equations closed under ordered paramodulation with respect to a well-founded ordering $\succ$. We also assume that the variables of every equation are disjoint (we rename, with fresh variables, the variables occurring in the conclusion of every inference).

Definition 7. *Let* $\succ_{\triangleright}$ *be an arbitrary west-ordering on* $T(\mathcal{F})$*, and* $\succ$ *the well-founded ordering used in the paramodulation rule. We define the ordering* $\succ_E$ *on* $\vec{E}$ *as follows: given two rules* $l \to r$ *and* $l' \to r'$ *in* $\vec{E}$*, we have* $l \to r \succ_E l' \to r'$ *if and only if* $l \succ_{\triangleright} l'$*, or* $l \equiv l'$ *and* $r \succ r'$*.*

Since $\succ_E$ is well-founded, we can now define, by induction on $\succ_E$, a ground TRS $\mathcal{R}$ *generated* by $\vec{E}$, and such that $\mathcal{R} \subseteq \vec{E}$. The *blockable-terms*, or simply $\mathcal{B}$-*terms*, are all terms that are right-hand sides of rules in $\mathcal{R}$.

Definition 8. *A rule e of the form $s \to t$ in $\vec{E}$* generates *the rule $s \to t$ if*

1. *s is irreducible by $\mathcal{R}^e$ at positions not in $P_{\mathcal{B}^e}(s)$ and*
2. *t is irreducible by $\mathcal{R}^e$ at positions not in $P_{\mathcal{B}^e}(t)$.*

where $\mathcal{R}^e$ is the set of rules generated by all rules e' in $\vec{E}$ such that $e \succ_E e'$, and $\mathcal{B}^e = Right(\mathcal{R}^e)$. We denote by $\mathcal{R}$ the set of rules generated by all rules in $\vec{E}$, and we define $\mathcal{B} = Right(\mathcal{R})$ as the set of $\mathcal{B}$-terms.

We call a $\mathcal{B}$-position *of t to any position in $P_{\mathcal{B}}(t)$ and a* non-$\mathcal{B}$-position *to any position of t not in $P_{\mathcal{B}}(t)$.*

We denote by $\mathcal{B}^{\lhd s}$ the set of terms t in $\mathcal{B}$ s.t. $s \rhd t$.

The way we define $\mathcal{R}$ is quite similar to the definition in [BG94] and especially the one in [BGNR99]. The main difference is that only non-$\mathcal{B}$-positions are required to be irreducible. As a consequence $\mathcal{R}$ can be non-terminating and overlapping. However, if the ordering $\succ$ is a west-ordering, i.e. it fulfils the subterm property as well, taking $\succ_{\rhd}$ as $\succ$, the generated $\mathcal{R}$ coincides with the rewrite system generated in [BGNR99]. Note that in this case the right-hand sides of $\mathcal{R}$ are irreducible, which makes the fact that we only consider non-$\mathcal{B}$-positions irrelevant.

Example 4. For the closed set of equations in Example 3, $\mathcal{R}$ consists of the following two rules:

$$\begin{array}{r}a \to g(a)\\ g(g(a)) \to g(a)\end{array}$$

Definition 9. *By $\vec{E}_{\mathcal{B}}$ we denote the subset of rules $l \to r$ in $\vec{E}$ s.t. $r \in \mathcal{B}$.*

Again following the example, $\vec{E}_{\mathcal{B}}$ includes the rules

$$\begin{array}{rl}a \to g(a) & \\ g^k(a) \to g(a) & \text{for all } k \geq 2\end{array}$$

The following lemmas state several relevant properties on $\mathcal{R}$, $\vec{E}_{\mathcal{B}}$ and $\mathcal{B}$, which will be used in the confluence proof, and in particular to show the so-called *return property*.

Lemma 4 roughly states that if a term is reducible (at non-$\mathcal{B}$-positions) at some point of the construction of $\mathcal{R}$ then it will keep irreducible from this point on. As a consequence all rules that have not been included in $\mathcal{R}$ are reducible at some non-$\mathcal{B}$-position by $\mathcal{R}$. In particular it happens for the left-hand sides of rules in $\vec{E}_{\mathcal{B}} \setminus \mathcal{R}$ (Lemma 5).

Lemma 4. *Let t be a term in $T(\mathcal{F})$ which is reducible by a rule $l \to r \in \mathcal{R}^e$ at positions not in $P_{\mathcal{B}^e}(t)$, for some rule e in $\vec{E}$. Then t is reducible by $l \to r$ at positions not in $P_{\mathcal{B}^d}(t)$ for all $d \succ_E e$ in $\vec{E}$.*

Proof. Suppose t is irreducible by $l \to r$ at positions not in $P_{\mathcal{B}^d}(t)$ for some $d \succ_E e$. Then there is some term $s \in \mathcal{B}^d \setminus \mathcal{B}^e$ with $t \trianglerighteq s \trianglerighteq l$. And there is a rule $e' = u \to s \in \mathcal{R}$ for some u, with $e' \succ_E e$. Wlog. we choose such an e' to be minimal wrt. $\succ_E$. Now, given that $u \to s \in \mathcal{R}$, we have that s is irreducible by $\mathcal{R}^{e'}$ at positions not in $P_{\mathcal{B}^{e'}}(s)$. We also have $s \trianglerighteq l$ and, moreover, $l \to r \in \mathcal{R}^{e'}$, since $e' \succ_E e$ and, hence, $\mathcal{R}^e \subseteq \mathcal{R}^{e'}$. Therefore, necessarily, there exists a term $w \in \mathcal{B}^{e'}$ with $w \trianglerighteq l$. And there is a rule $e'' = v \to w \in \mathcal{R}^{e'}$. But then $e' \succ_E e''$, and this contradicts the minimality of e'. □

Lemma 5. *If $l \to r$ is a rule in $\vec{E}_{\mathcal{B}} \setminus \mathcal{R}$ then l is reducible by $\mathcal{R}$ at some non-$\mathcal{B}$-position.*

Proof. We proceed by contradiction. Let $e = l \to r$ be a rule in $\vec{E}_{\mathcal{B}} \setminus \mathcal{R}$. Suppose l is irreducible by $\mathcal{R}$ at non-$\mathcal{B}$ positions. Then l is irreducible by $\mathcal{R}^e$ at non-$\mathcal{B}$-positions, since $\mathcal{R}^e \subseteq \mathcal{R}$. Moreover l is irreducible by $\mathcal{R}^e$ at positions not in $P_{\mathcal{B}^e}(l)$: otherwise l would be reducible by $\mathcal{R}^e$ at positions not in $P_{\mathcal{B}^d}(l)$ for all $d \succ_E e$ by Lemma 4, contradicting the assumption. Now note that $e \in \vec{E}$, since $\vec{E}_{\mathcal{B}} \subseteq \vec{E}$. Therefore, by definition of $\mathcal{R}$, if l is irreducible by $\mathcal{R}^e$ at positions not in $P_{\mathcal{B}^e}(l)$ and $e \notin \mathcal{R}$ then we must have r is reducible by $\mathcal{R}^e$ at some position not in $P_{\mathcal{B}^e}(r)$. Then, by Lemma 4, we have r is reducible by $\mathcal{R}^e$ at some position not in $P_{\mathcal{B}^d}(r)$ for all $d \succ_E e$. Therefore r is reducible by $\mathcal{R}$ at some position not in $P_{\mathcal{B}}(r)$. But then we have a contradiction, since $e \in \vec{E}_{\mathcal{B}}$ implies $r \in \mathcal{B}$ and hence the set of positions not in $P_{\mathcal{B}}(r)$ is empty. □

¿From these two previous lemmas we can infer that $\mathcal{B}$-terms do not appear neither as left-hand sides of rules in $\mathcal{R}$ nor as left-hand sides of rules in $\vec{E}_{\mathcal{B}}$.

Lemma 6. *Let t be a term in $\mathcal{B}$. Then $t \notin Left(\mathcal{R})$.*

Proof. By induction on $\succ_{\triangleright}$. If $t \in \mathcal{B}$ then there is a rule $e = s \to t \in \mathcal{R}$. Wlog. we choose such an e with t as right-hand side to be minimal wrt. $\succ_E$. Then $t \notin \mathcal{B}^e$ and, therefore, $\lambda \notin P_{\mathcal{B}^e}(t)$. On the other hand, $s \to t \in \mathcal{R}$ implies t is irreducible by $\mathcal{R}^e$ at positions not in $P_{\mathcal{B}^e}(t)$ and, given that $\lambda \notin P_{\mathcal{B}^e}(t)$, we have $t \notin Left(\mathcal{R}^e)$. Moreover, we have $s \neq t$, since $s \succ t$. To conclude, we show that there is no rule $e' = t \to u \in \mathcal{R}$ with $e' \succ_E e$, i.e. that no greater rule than e in $\mathcal{R}$ can have t as left-hand side. Since $s \neq t$, we just have to consider the case where $t \succ_{\triangleright} s$. If $s \to t$ and $t \to u$ are two rules in $\mathcal{R}$, then there are two equations $s' \simeq t'$ and $t'' \simeq u'$ in E s.t., for some ground substitution σ, we have $s'\sigma \equiv s$, $t'\sigma \equiv t''\sigma \equiv t$ and $u'\sigma \equiv u$. Two cases have to be considered:

1. $t' \notin \mathcal{X}$. Then there is an inference by ordered paramodulation

$$\frac{t'' \simeq u' \qquad s' \simeq t'}{(s' \simeq u')\theta}$$

where $\theta = mgu(t', t'')$. Note that, wlog., we can choose σ s.t. $(s' \simeq u')\theta\sigma \equiv s'\sigma \simeq u'\sigma$. Moreover, note that $s \simeq u$ is not a taulotogy of the form $w \simeq w$, since $s \succ t \succ u$. Then $s \simeq u \in E$ and $s \to u \in \vec{E}$.

Now note that $s \to t \succ_E s \to u$, since $t \succ u$. And, as stated before, $t \to u \succ_E s \to t$. Moreover, by induction hypothesis, we have that $s \notin \mathcal{B}$, since $t \succ_\triangleright s$ and $s \to t \in \mathcal{R}$. Therefore $s \to u$ has not generated rule in $\mathcal{R}$: otherwise $s \to t$ would have failed condition 1 to generate rule in $\mathcal{R}$. Now we analyse the possible reasons why $e'' = s \to u$ has not generated rule:

(a) s is reducible by $\mathcal{R}^{e''}$ at positions not in $P_{\mathcal{B}^{e''}}(s)$. Then, by Lemma 4, s is reducible by $\mathcal{R}^{e''}$ (and hence by $\mathcal{R}^{e}$) at positions not in $P_{\mathcal{B}^{e}}(s)$, contradicting $s \to t \in \mathcal{R}$.

(b) u is reducible by $\mathcal{R}^{e''}$ at positions not in $P_{\mathcal{B}^{e''}}(u)$. Then, by Lemma 4, u is reducible by $\mathcal{R}^{e''}$ (and hence by $\mathcal{R}^{e'}$) at positions not in $P_{\mathcal{B}^{e'}}(u)$, contradicting $t \to u \in \mathcal{R}$.

2. $t' = x \in \mathcal{X}$. In this case we have $x \notin vars(s')$, given that $t \succ_\triangleright s$ implies $s \not\trianglerighteq t$. Then let σ' be a substitution s.t. $x\sigma' \equiv u$ and $s'\sigma \equiv s\sigma$. We have $(s' \simeq t')\sigma' \equiv s \simeq u$. Then $s \to u \in \vec{E}$, since $s \succ u$. As before we have $t \to u \succ_E s \to t \succ_E s \to u$ and the proof is analogous.

□

Lemma 7. *Let t be a term in $\mathcal{B}$. Then $t \notin Left(\vec{E}_\mathcal{B})$.*

Proof. We proceed by contradiction. Assume t is a term in $\mathcal{B}$ and there is some rule of the form $t \to u \in \vec{E}_\mathcal{B}$. Since t is a term in $\mathcal{B}$ we have $t \notin Left(\mathcal{R})$ by Lemma 6. Then $t \to u \in \vec{E}_\mathcal{B} \setminus \mathcal{R}$. But then by Lemma 5 we have t is reducible by $\mathcal{R}$ at some non-$\mathcal{B}$-position, contradicting $t \in \mathcal{B}$ since all positions of a $\mathcal{B}$-term are $\mathcal{B}$-positions. □

Therefore it follows that there are no two different rules in $\vec{E}_\mathcal{B}$ with the same left-hand side (by showing contradiction with the previous lemma after applying a paramodulation step between the left-hand sides of both rules).

Lemma 8. *If $l \to r$ is a rule in $\vec{E}_\mathcal{B}$ then l is irreducible by $\vec{E}_\mathcal{B} \setminus \{l \to r\}$ at topmost position.*

Proof. We proceed by contradiction. Assume $l \to r$ and $l \to s$ are two rules in $\vec{E}_\mathcal{B}$ with $r \succ s$. Then there are two equations $l' \simeq r'$ and $l'' \simeq s'$ in E such that for some substitution σ we have $l'\sigma \equiv l$, $r'\sigma \equiv r$, $l''\sigma \equiv l$ and $s'\sigma \equiv s$. The following cases are considered:

1. $l' \notin \mathcal{X}$. Then there is an inference by ordered paramodulation

$$\frac{l'' \simeq s' \qquad l' \simeq r'}{(s' \simeq r')\theta}$$

where $\theta = mgu(l', l'')$, whose conclusion has an instance of the form $s \simeq r$. Then $r \to s \in \vec{E}_\mathcal{B}$ since $r \succ s$. But this contradicts Lemma 7 since $r \in \mathcal{B}$.

2. $l' = x \in \mathcal{X}$ and $l'' \notin \mathcal{X}$. Then there is an inference by ordered paramodulation

$$\frac{l' \simeq r' \qquad l'' \simeq s'}{(r' \simeq s')\theta}$$

where $\theta = mgu(l', l'') = \{x \mapsto l''\}$, whose conclusion has an instance of the form $r \simeq s$. Then we conclude as before.

3. $l' = x \in \mathcal{X}$, $l'' = y \in \mathcal{X}$ and $y \in vars(s')$. Then we proceed exactly as in case 2.
4. $l' = x \in \mathcal{X}$, $l'' = y \in \mathcal{X}$ and $y \notin vars(s')$. Then let σ' be a substitution such that $y\sigma' \equiv r$ and $z\sigma' \equiv z\sigma$ for all other variables z. We have $l''\sigma' \equiv r$ and $s'\sigma' \equiv s$, and we conclude as in previous cases.

□

The following is an easy consequence of Lemma 5.

Lemma 9. *Let t be a term in $T(\mathcal{F})$. If t is reducible by $\vec{E}_{\mathcal{B}}$ at some non-$\mathcal{B}$-position p then t is reducible by $\mathcal{R}$ at p or at some non-$\mathcal{B}$-position below p.*

Proof. Let $l \to r$ be a rule in $\vec{E}_{\mathcal{B}}$ reducing t at some non-$\mathcal{B}$-position p. If $l \to r \in \mathcal{R}$ then we are done. If, otherwise, $l \to r \notin \mathcal{R}$ then l is reducible by $\mathcal{R}$ at some non-$\mathcal{B}$-position p' by Lemma 5. But then p' is also a non-$\mathcal{B}$-position of t, since p is a non-$\mathcal{B}$-position of t. □

Finally all these previous properties let us prove the following property, which precisely[1] characterizes $\mathcal{R}$ in terms of $\vec{E}_{\mathcal{B}}$ and $\mathcal{B}$.

Lemma 10. *Let $l \to r$ be a rule in $\vec{E}_{\mathcal{B}}$. Then $l \to r \in \mathcal{R}$ if and only if l is irreducible by $\vec{E}_{\mathcal{B}}$ at any non-$\mathcal{B}$-position apart from λ.*

Proof. We first prove the left-to-right implication by contradiction. Let $e = l \to r$ be a rule in $\mathcal{R}$. Suppose l is reducible by $\vec{E}_{\mathcal{B}}$ at some non-$\mathcal{B}$-position $p \neq \lambda$. Then l is reducible by some rule $e' = l' \to r'$ in $\mathcal{R}$ at position p or at some non-$\mathcal{B}$-position below p, by Lemma 9. Let p' denote this position. Moreover we have $e \succ_E e'$, since $p \neq \lambda$ and, hence, $l \rhd l'$. Therefore l is reducible by $\mathcal{R}^e$ at some non-$\mathcal{B}$-position. Then, in particular, it is reducible by $\mathcal{R}^e$ at some position not in $P_{\mathcal{B}^e}$. But this contradicts $l \to r \in \mathcal{R}$.

Wrt. the right-to-left implication, by Lemma 8 we have that the only rule in $\vec{E}_{\mathcal{B}}$ reducing l at topmost position is precisely $l \to r$. Now recall that $\mathcal{R} \subseteq \vec{E}_{\mathcal{B}}$. So we have that if l is reducible by $\mathcal{R}$ at topmost position then $l \to r \in \mathcal{R}$. And if, otherwise, l is irreducible by $\mathcal{R}$ at topmost position then l is irreducible by $\mathcal{R}$ at any non-$\mathcal{B}$-position. But then, by Lemma 5, we have $l \to r \notin \vec{E}_{\mathcal{B}} \setminus \mathcal{R}$, which implies $l \to r \in \mathcal{R}$ given that $l \to r \in \vec{E}_{\mathcal{B}}$. Therefore in either case we have $l \to r \in \mathcal{R}$. □

This property generalizes the notion in [BGNR99] of irreducibility of rules of the ground TRS that defines the model. Here irreducibility amounts to irreducibility at non-$\mathcal{B}$-positions of the left and right-hand sides. Note that, in particular, if the right-hand sides of rules in $\mathcal{R}$ are irreducible by $\mathcal{R}$, then left-hand sides are irreducible as well, and both notions of irreducibility coincide.

4.1 The return property

In this section we prove the so-called *return property* which roughly states that if we reduce a $\mathcal{B}$-term t into a term u then there is a rewrite sequence from

[1] Note that, by definition of $\mathcal{R}$ and $\vec{E}_{\mathcal{B}}$, we have $\mathcal{R} \subseteq \vec{E}_{\mathcal{B}}$.

u to t again (without reducing any $\mathcal{B}$-subterm). This property is crucial to prove that $\mathcal{R}$ is confluent.

The intuition behind this property (at the ground level) comes from the fact that t is minimal wrt. $\succ$, and that moreover ordered paramodulation gives us $t_1 \simeq t_n$ for any chain $t_1 \overset{\succ}{\simeq} t_2 \cdots \overset{\succ}{\simeq} t_n$ of equations in the closure. Then, if t_n is minimal wrt. $\succ$ and it is reducible by some some rule, for instance $t_1 \to t_n$ itself, at some position $p \neq \lambda$ (which may happen since $\succ$ can contradict the subterm property), we can obtain $t_1 \simeq t_n[t_n]_p$ by an ordered paramodulation step, which gives us that $t_n[t_n]_p$ is in the same chain of equations and, hence, $t_n[t_n]_p \overset{\succ}{\simeq} t_n$ is in the closure as well. In the example, since we have $a \overset{\succ}{\simeq} g(a)$ with $g(a)$ minimal and $g(a)$ reducible by $a \to g(a)$, we obtain $a \simeq g(g(a))$ and hence $g(g(a)) \overset{\succ}{\simeq} g(a)$. Thus if we reduce $g(a)$ with $a \to g(a)$, we can return to $g(a)$ using $g(g(a)) \to g(a)$.

Thanks to this return property, t_n can be chosen as a representative (not a normal form, since it can be reducible) of all $t_1, \ldots, t_n$. Moreover for any of these terms t_i the representative can be obtained by an innermost rewrite sequence consisting in replacing any reducible subterm by its representative. These representatives are the so-called $\mathcal{B}$-terms.

As usual, the proof becomes rather technical due to the reductions applied at positions that are below a variable in the equation in E which has generated the rule $l \to t_n$ in $\mathcal{R}$ (recall that the $\mathcal{B}$-terms are the right-hand sides of rules in $\mathcal{R}$). For this reason, many times we have to reduce in one shot all redexes involving the same subterm. This is denoted by a kind of parallel rewriting.

Now, we will give the main definitions and lemmas which allow us to prove the return property (all proofs can be found in [BR02]). First we define the set of reducible positions with respect to a given reduction relation.

Definition 10. *Let R be a TRS. Let $\curvearrowright$ denote a binary relation on abstracted terms wrt. R. And let $s \cdot \gamma$ be an abstracted term wrt. R. Then $red\_pos(s \cdot \gamma, \curvearrowright)$ denotes the set of positions $\bigcup_{t \cdot \delta} P_{\mathcal{X}}(t) \setminus P_{\mathcal{X}}(s)$ for all $t \cdot \delta$ such that $s \cdot \gamma \curvearrowright t \cdot \delta$.*

Definition 12 establishes a kind of innermost rewrite notion, which fairly mimics the aforementioned process for obtaining the representative of a given term. Since this special rewriting strategy is intended to be used in the proof of the return property, we are obliged to define it not only as an innermost, but also as a kind of parallel rewrite notion (Definition 11) in order to express substitution replacements at the non-ground level. Fortunately, due to the parallel rewriting strategy, any *illegal* rewrite subsequence (that is, a rewrite subsequence at or below a $\mathcal{B}$-position) can be omitted by applying the return property by induction on smaller $\mathcal{B}$-subterms (this is mainly what Lemma 15 states).

Definition 11. *Let R be a TRS with non-overlapped left-hand sides. Let $\curvearrowright$ denote a binary relation on abstracted terms wrt. R, such that $\curvearrowright \subseteq \curvearrowright_R$. We write $s \cdot \gamma \overset{\parallel}{\curvearrowright} t \cdot \delta$ if there is a rewrite sequence D of the form $s \cdot \gamma \overset{+}{\curvearrowright} t \cdot \delta$ and $t|_p \in \mathcal{X}$ for all $p \in red\_pos(s \cdot \gamma, \curvearrowright)$.*

Note that requiring $t|_p \in \mathcal{X}$ implies p to be a valid position in t and hence the rewrite steps not being overlapped.

By definition, any rewrite step with $\overset{\parallel}{\curvearrowright}$ represents a rewrite sequence with $\curvearrowright$.

Definition 12. *Let R be a TRS. Let $u \cdot \gamma$ be an abstracted term wrt. R. We write $u \cdot \gamma \rightsquigarrow_R v \cdot \delta$ if $u \cdot \gamma \overset{\parallel}{\curvearrowright}_{\underline{\{l \to r\}}} v \cdot \delta$ with $l \to r \in R$, $v \notin \mathcal{X}$, and there is no rule $s \to t \in R$ with $l \rhd s$ such that $u \cdot \gamma \curvearrowright_{\underline{\{s \to t\}}} w \cdot \rho$.*

Again, by definition, any non-empty rewrite sequence with $\rightsquigarrow_R$ (i.e. $u \cdot \gamma \overset{+}{\rightsquigarrow}_R w \cdot \rho$) represents a rewrite sequence with $\curvearrowright_R$ (i.e. $u \cdot \gamma \overset{+}{\curvearrowright}_R w \cdot \rho$). If D is a rewrite sequence which includes steps with $\curvearrowright_R$ and $\rightsquigarrow_R$ for some TRS R, we will write $D(\curvearrowright_R)$ to denote the corresponding rewrite sequence with only $\curvearrowright_R$ steps.

Lemma 11 is a rather simple but useful lemma, since it allows us to guarantee that during the transformation of a $\mathcal{B}$-term t towards itself in the proof of the return property, no tautology of the form $t \simeq t$ is obtained at the equational level till the last step. It is very important to show that we get back to t at the last step, since then all *illegal* steps on smaller $\mathcal{B}$-subterms are guaranteed to be already *repaired* and hence they can be omitted, as Lemma 15 states.

Lemma 11. *Let t be a term in $T(\mathcal{F})$ and p a position not in $P_{\mathcal{B}^{\lhd t}}(t)$. If $t \curvearrowright_{\vec{E}_{\mathcal{B}}} t[x]_p \cdot \delta \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t' \cdot \gamma$ and $t' \notin \mathcal{X}$ then $t'\gamma \not\equiv t$.*

Proof. If $t \curvearrowright_{\vec{E}_{\mathcal{B}}} t[x]_p \cdot \delta \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t' \cdot \gamma$, $t' \notin \mathcal{X}$ and $t'\gamma \equiv t$ then $p \in P_{\mathcal{X}}(t', \gamma)$ and, hence, $p \in P_{\mathcal{B}}(t'\gamma)$. More precisely, we have $p \in P_{\mathcal{B}^{\lhd t'\gamma}}(t'\gamma)$, since $t' \notin \mathcal{X}$. But then $p \in P_{\mathcal{B}^{\lhd t}}(t)$, contradicting the assumption. $\square$

The following two lemmas roughly state that the innermost rewriting strategy used to prove the return property leads to rewrite sequences which are unique up to permutation of non-overlapped steps (although this is not a crucial property for the proof, it somehow simplifies the reasonings).

Lemma 12. *Let R be a TRS with non-overlapped left-hand sides. Then any term s in $T(\mathcal{F}, \mathcal{X})$ has exactly one normal form $s_n \cdot \gamma$ wrt. $\curvearrowright_{\overline{R}}$, and all the rewrite sequences with $\curvearrowright_{\overline{R}}$ of $s_n \cdot \gamma$ from s are a permutation of the same steps.*

Proof. Since rules in R have non-overlapped left-hand sides and, by definition of $\curvearrowright_{\overline{R}}$, in a rewrite sequence with $\curvearrowright_{\overline{R}}$ there are no steps at overlapped positions, then we can rearrange the steps of any rewrite sequence D with $\curvearrowright_{\overline{R}}$ leading to a normal form from s and nothing else but a permutation of the steps in D can lead to a normal form wrt. $\curvearrowright_{\overline{R}}$ of s from s. $\square$

Lemma 13. *Let $u \cdot \gamma$ be an abstracted term wrt. $\vec{E}_{\mathcal{B}}$. Then for any term $v \cdot \delta$ there is at most one possible rewrite sequence $u \cdot \gamma \overset{*}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} v \cdot \delta$ up to permutation of non-overlapped steps.*

Proof. It follows from the innermost rewriting strategy of $\rightsquigarrow_{\vec{E}_{\mathcal{B}}}$ (see definition 12) and from the fact that there are no two different rules in $\vec{E}_{\mathcal{B}}$ with the same left-hand side by Lemma 8. $\square$

Now we show that, since our rewriting strategy is an innermost strategy, the steps which are applied at or below $\mathcal{B}$-positions different from λ can be omitted, by using the return property (for rewrite sequences using $\vec{E}_{\mathcal{B}}$, see Lemma 26) by induction on smaller $\mathcal{B}$-subterms. We split the proof in two lemmas.

Lemma 14. *Let t be a term in $\mathcal{B}$. Let $\mathcal{S}$ be a subset of $\vec{E}_{\mathcal{B}}$ with non-overlapped left-hand sides. Let D be a rewrite sequence of the form $t \overset{!}{\curvearrowright}_{\overline{\mathcal{S}}} t_1 \cdot \rho \overset{!}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} t_m \cdot \beta$. Then there is a rewrite sequence D' of the form $t \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t_n \cdot \alpha \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t'_m \cdot \beta'$ which is equal to $D(\curvearrowright_{\vec{E}_{\mathcal{B}}})$ up to permutation of non-overlapped steps, and such that $t_n\alpha \equiv t$ and $t_n \cdot \alpha \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t'_m \cdot \beta'$ consists of the steps in $D(\curvearrowright_{\vec{E}_{\mathcal{B}}})$ which are applied at positions not in $P_{\mathcal{B}^{\lhd t}}(t)$.*

Proof. We assume Lemma 26 is true for terms whose size is smaller than the size of t. Recall that D is unique (up to permutation of non-overlapped steps) by lemmas 12 and 13, and hence so is $D(\curvearrowright_{\vec{E}_{\mathcal{B}}})$. By definition of $\curvearrowright_{\vec{E}_{\mathcal{B}}}$, no step in $D(\curvearrowright_{\vec{E}_{\mathcal{B}}})$ is applied at or below an already rewrited position, that is, any position p becomes blocked when a step is applied at such position, and subsequent steps are applied only above p or at some position which is not overlapped with p. Besides, the applicability of any step does not depend on the context over the position in which it is applied. Therefore the steps in $D(\curvearrowright_{\vec{E}_{\mathcal{B}}})$ can be rearranged so that all steps at positions in $P_{\mathcal{B}^{\lhd t}}(t)$ are applied before than others. That is, there exists a rewrite sequence D' which is equal to $D(\curvearrowright_{\vec{E}_{\mathcal{B}}})$ up to permutation of non-overlapped steps and such that it is of the form $t \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t_n \cdot \alpha \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t'_m \cdot \beta'$, where $t_n \cdot \alpha \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t'_m \cdot \beta'$ consists of the steps in $D(\curvearrowright_{\vec{E}_{\mathcal{B}}})$ which are applied at positions not in $P_{\mathcal{B}^{\lhd t}}(t)$. We conclude showing that $t_n \cdot \alpha \equiv t$. If no rule in $\mathcal{S}$ reduces t at some position in $P_{\mathcal{B}^{\lhd t}}(t)$ then we are done. Otherwise let P denote the set of positions p in $P_{\mathcal{X}}(arg\_abs(t, \mathcal{R}))$ such that $\mathcal{S}$ reduces $t|_p$. Note that the innermost rewriting strategy of the rewrite sequence D' on t implies an innermost rewriting strategy on $t|_p$ for any $p \in P$. And, by Lemma 26, we have that if $t|_p \overset{!}{\curvearrowright}_{\overline{\mathcal{S}}} t' \cdot \gamma \overset{!}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} t'' \cdot \delta$ then $t''\delta \equiv t|_p$ or $t''\delta \to t|_p \in \vec{E}_{\mathcal{B}}$. Observe that in either case no further step is applied at or below p in D', since $t'' \cdot \delta$ is irreducible wrt. $\rightsquigarrow_{\vec{E}_{\mathcal{B}}}$. Moreover, if $t''\delta \equiv t|_p$ then no further step can be applied exactly at position p in D' since, by Lemma 7, we have $t|_p \notin Left(\vec{E}_{\mathcal{B}})$. If, otherwise, we have $t''\delta \to t|_p \in \vec{E}_{\mathcal{B}}$ then this rule must have been applied at position p in D' in some step before $t_n \cdot \alpha$ is obtained, since $p \neq \lambda$ and there are no two different rules in $\vec{E}_{\mathcal{B}}$ with the same left-hand side by Lemma 8. Furthermore, as stated before, no rule is applied above p before $t_n \cdot \alpha$ is obtained. Then we have $t_n \cdot \alpha \equiv t$. $\square$

Lemma 15. *Let t be a term in $\mathcal{B}$. Let $\mathcal{S}$ be a subset of $\vec{E}_{\mathcal{B}}$ with non-overlapped left-hand sides. Let D be a rewrite sequence of the form $t \overset{!}{\curvearrowright}_{\overline{\mathcal{S}}} t_1 \cdot \rho \overset{!}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} t_m \cdot \beta$.*

Then there is a rewrite sequence D' of the form $arg\_abs(t,\mathcal{R}) \overset{*}{\curvearrowright}_{\vec{E}_\mathcal{B}} t'_m \cdot \beta'$ with $t'_m\beta' \equiv t_m\beta$ consisting of the steps in $D(\curvearrowright_{\vec{E}_\mathcal{B}})$ which are applied at positions not in $P_{\mathcal{B}^{\triangleleft t}}(t)$.

Proof. By Lemma 14 we have there is a rewrite sequence D'' of the form $t_n \cdot \alpha \overset{*}{\curvearrowright}_{\vec{E}_\mathcal{B}} t'_m \cdot \beta'$ consisting of the steps in $D(\curvearrowright_{\vec{E}_\mathcal{B}})$ which are applied at positions not in $P_{\mathcal{B}^{\triangleleft t}}(t)$ and where $t_n\alpha \equiv t$. Now let D' be the rewrite sequence consisting of the same steps as in D'' but beginning with $arg\_abs(t,\mathcal{R})$ instead of $t_n \cdot \alpha$. This is a rewrite sequence with $\curvearrowright_{\vec{E}_\mathcal{B}}$, since no steps are applied at positions in $P_{\mathcal{B}^{\triangleleft t}}(t)$ and hence the blocked positions in $arg\_abs(t,\mathcal{R})$ do not forbid any step. Moreover the last term in D'' is equivalent to the last term in $D(\curvearrowright_{\vec{E}_\mathcal{B}})$, and so is the last term in D'. □

Lemma 26 establishes the return property of $\mathcal{B}$-terms for rewrite sequences using $\vec{E}_\mathcal{B}$. Its proof, although based on the relatively simple intuition given at the beginning of this subsection, is quite long and technical, due mainly to the parallel rewiting issue combined with the need of explicity constructing a fair and complete return path to the initial term while avoiding tautologies.

We must point out that, although our aim is to prove the return property for rewrite sequences using $\mathcal{R}$ (Theorem 1), this weaker lemma is necessary since sometimes illegal steps can only be repaired by using rules in $\vec{E}_\mathcal{B} \setminus \mathcal{R}$.

The following are two auxiliar lemmas used in subsequent proofs.

Lemma 16. *Let t be a term in $T(\mathcal{F})$ and R a TRS such that $l \not\equiv r$ for all $l \to r \in R$. If $t \overset{+}{\curvearrowright}_{\overline{R}} u \cdot \delta$ then for any position $p \in P_\mathcal{X}(u)$ we have $t|_p \not\equiv u\delta|_p$.*

Proof. Since $l \not\equiv r$ for all $l \to r \in R$ and, moreover, in a rewrite sequence by $\overset{+}{\curvearrowright}_{\overline{R}}$ there are no overlapped rewriting steps, it follows that $t|_p \not\equiv u\delta|_p$ for any rewrited position p. □

Lemma 17. *Let t be a term in $\mathcal{B}$ and $s \to t$ the minimal rule wrt. $\succ_E$ in $\mathcal{R}$ with t as right-hand side. If a rule $l \to r$ in $\mathcal{R}$ reduces t at some position not in $P_{\mathcal{B}^{\triangleleft t}}(t)$ then $l \to r \succeq_E s \to t$.*

Proof. If $e = s \to t$ is the minimal rule wrt. $\succ_E$ in $\mathcal{R}$ with t as right-hand side then $t \notin \mathcal{B}^e$ and, hence, $P_{\mathcal{B}^e}(t) \subseteq P_{\mathcal{B}^{\triangleleft t}}(t)$. Then $l \to r$ reduces t at some position not in $P_{\mathcal{B}^e}(t)$. Now assume $s \to t \succ_E l \to r$. Then $l \to r \in \mathcal{R}^e$ and, by Lemma 4, t is reducible by $l \to r$ at positions not in $P_{\mathcal{B}^d}(t)$ for all $d \succ_E e$, contradicting $t \in \mathcal{B}$. □

Here follows the most technical part of the proof of the return property. Lemmas 18 to 26, together with lemmas 14 and 15, constitute an split proof of the return property for rewrite sequences using $\vec{E}_\mathcal{B}$.

Definition 13. *Let t be a term in $T(\mathcal{F},\mathcal{X})$, σ a ground substitution and $t' \cdot \gamma$ an abstraction of $t\sigma$ wrt. $\vec{E}_\mathcal{B}$. We say that a valid position p in t is* above a variable *of t' p is a proper prefix of some position in $P_\mathcal{X}(t')$.*

Definition 14. *Let t be a term in $T(\mathcal{F},\mathcal{X})$, σ a ground substitution and $t' \cdot \gamma$ an abstraction of $t\sigma$ wrt. $\vec{E}_\mathcal{B}$.*

1. *By* $yellow\_vars(t,t')$ *we denote the set of variables* x *in* $vars(t)$ *such that* $t|_p \equiv x$ *for some position* p *above a variable of* t'.
2. *By* $green\_vars(t,t')$ *we denote the set of variables* x *in* $vars(t)$ *such that* $t|_p \equiv x$ *for some position* p *which is not above a variable of* t'.

Note that we have $green\_vars(t,t') \cup yellow\_vars(t,t') = vars(t)$. *Moreover,* $green\_vars(t,t') \cap yellow\_vars(t,t')$ *may not be empty.*

Definition 15. *Let* R *be a TRS. Let* $\curvearrowright$ *denote a binary relation on abstracted terms wrt.* R. *Let* t *be a term in* $T(\mathcal{F},\mathcal{X})$, σ *a ground substitution and* $t' \cdot \gamma$ *an abstraction of* $t\sigma$ *wrt.* R. *By* $red\_vars(t, t' \cdot \gamma, \curvearrowright)$ *we denote the set of variables* x *in* $yellow\_vars(t,t')$ *such that* $t|_p \equiv x$ *for some prefix* p *of some position in* $red\_pos(t' \cdot \gamma, \curvearrowright)$.

Definition 16. *Let* t *be a term in* $\mathcal{B}$. *We say that* t *is* self-returning *for some rule* $l \to r$ *in* $\vec{E}_{\mathcal{B}}$ *if the minimal rule wrt.* $\succ_E$ *in* $\mathcal{R}$ *with* t *as right-hand side is of the form* $C[r]_p \to C[l]_p$ *for some position* p *not in* $P_{\mathcal{B}^{\triangleleft t}}(t)$ *and there is some equation* $s' \simeq t'$ *in* E *such that, for some substitution* σ, *we have* $s'\sigma \equiv C[r]_p$, $t'\sigma \equiv C[l]_p$ *and* p *is a valid non-variable position of* t'.

Lemma 18. *Let* t *be a term in* $\mathcal{B}$. *Let* $\mathcal{S}$ *be a non-empty subset of* $\vec{E}_{\mathcal{B}}$ *with non-overlapped left-hand sides and such that each rule in* $\mathcal{S}$ *reduces* t *at some position not in* $P_{\mathcal{B}^{\triangleleft t}}(t)$. *If* t *is non-self-returning for any rule in* $\mathcal{S}$ *then there is an equation* $u \simeq v \in E$ *such that, for some substitution* φ, *we have*

1. $u\varphi \equiv t$,
2. $v\varphi \equiv t_0\delta$ *with* $t \overset{+}{\curvearrowright}_{\overline{\mathcal{S}}} t_0 \cdot \delta$,
3. $\forall q \in red\_pos(t_0 \cdot \delta, \curvearrowright_{\overline{\mathcal{S}}}) : q$ *is a valid non-variable position in* v *and*
4. $yellow\_vars(v,t_0) \cap (vars(u) \cup green\_vars(v,t_0)) = \emptyset$.

Proof. Let $s \to t$ be the minimal rule wrt. $\succ_E$ in $\mathcal{R}$ with t as right-hand side, and $s' \simeq t'$ an equation in E such that $s'\sigma \equiv s$ and $t'\sigma \equiv t$ for some substitution σ. Let V denote the set of variables x in $vars(t')$ such that $x\sigma$ is reducible by $\mathcal{S}$. Several cases have to be considered:

1. $V = \emptyset$. Let $l \to r$ be a rule in $\mathcal{S}$ (recall that $\mathcal{S}$ is not empty) reducing t at some position p not in $P_{\mathcal{B}^{\triangleleft t}}(t)$, and $l' \simeq r'$ an equation in E such that (wlog.) $l'\sigma \equiv l$ and $r'\sigma \equiv r$. Then there is an inference

$$\frac{l' \simeq r' \qquad s' \simeq t'}{(s' \simeq t'[r']_p)\theta}$$

where $\theta = mgu(t'|_p, l')$, whose conclusion has an instance of the form $s \simeq t[r]_p$ under the substitution σ. Note that, since $V = \emptyset$, p is a non-variable position of t'. And we have $t \curvearrowright_{\overline{\mathcal{S}}} t[x]_p \cdot \{x \mapsto r\}$ for some variable x. Moreover $s \not\equiv t[r]_p$, since otherwise t would be self-returning (see definition 16) for $l \to r$. Let t_0 denote $t[x]_p$ and δ denote $\{x \mapsto r\}$.

(a) $s'\theta = y \in \mathcal{X} \wedge y \notin vars(t'[r']_p\theta)$. Then let φ be a substitution such that $y\varphi \equiv t$ and $z\varphi \equiv z\sigma$ for all other variables z. And let u denote $s'\theta$ and v denote $t'[r']_p\theta$. We have that $u \simeq v$ fulfils the required conditions under the substitution φ:
   - $u\varphi \equiv t$, since $u = y$.
   - $v\varphi \equiv t_0\delta$ with $t \curvearrowright_{\overline{\mathcal{S}}} t_0 \cdot \delta$. This is true since $y \notin vars(v)$, and therefore $v\varphi \equiv v\sigma$.
   - $\forall q \in red\_pos(t_0 \cdot \delta, \curvearrowright_{\overline{\mathcal{S}}}) : q$ is a valid non-variable position in v. This is true since $V = \emptyset$ and $v[\,]_p$ is a particularization of $t'[\,]_p$ (note that $p \notin red\_pos(t_0 \cdot \delta, \curvearrowright_{\overline{\mathcal{S}}})$).
   - $yellow\_vars(v, t_0) \cap (vars(u) \cup green\_vars(v, t_0)) = \emptyset$. We have $t'|_p$ is not a variable and hence $yellow\_vars(v, t_0) = \emptyset$.

(b) $s'\theta \notin \mathcal{X} \vee s'\theta = y \in vars(t'[r']_p\theta)$. Then there is an inference

$$\frac{s' \simeq t' \qquad s'\theta \simeq t'[r']_p\theta}{(t' \simeq t'[r']_p\theta)\gamma}$$

where $\gamma = mgu(s', s'\theta)$, whose conclusion has an instance of the form $t \simeq t[r]_p$ under the substitution σ, which is clearly not a tautology of the form $w \simeq w$. Note that this is a correct inference since, if $s'\theta$ is a variable y, then $y \in vars(t'[r']_p\theta)$ and s' is also a variable. Now we choose $t'\gamma$ as u, $t'[r']_p\theta\gamma$ as v and σ as φ, and we have $u\varphi \equiv t$, $v\varphi \equiv t_0\delta$ with $t \curvearrowright_{\overline{\mathcal{S}}} t_0 \cdot \delta$, and the remaining conditions are fulfiled for the same reasons as in case 1a.

$V \neq \emptyset$. We have $s' \not\trianglerighteq y$ (but possibly $s' = y$) for all $y \in V$: we have $y\sigma \trianglerighteq l$ for some rule $l \to r$ in $\mathcal{S}$ and hence in $\vec{E}_{\mathcal{B}}$ which reduces t at some position not in $P_{\mathcal{B}^{\triangleleft t}}(t)$. Then, by Lemma 9, there is some rule $l_1 \to r_1$ in $\mathcal{R}$ with $l \trianglerighteq l_1$ which reduces t at some position not in $P_{\mathcal{B}^{\triangleleft t}}(t)$. And, by Lemma 17, we have $l_1 \to r_1 \succeq_E s \to t$, and hence $s \not\trianglerighteq l_1$, which implies $s' \not\trianglerighteq y$. Then let σ' be a substitution such that $\forall y \in V : y\sigma' \equiv y\sigma!_{\curvearrowright_{\overline{\mathcal{S}}}}$ and $z\sigma' \equiv z\sigma$ for all other variables z. Two cases have to be considered:

1. $s'\sigma' \not\equiv s$. Then we necessarily have $s' = y$ for some variable y in V and hence in $vars(t')$. Now consider any rule $l \to r$ in $\mathcal{S}$ which reduces $y\sigma$. We have $l \to r \succeq_E s \to t$ by Lemma 17. Then necessarily $l \equiv s$, since $s \trianglerighteq l$. But then $l \to r = s \to t$ by Lemma 8. Therefore we have $s \to t \in \mathcal{S}$, and hence $s'\sigma' \equiv t$. In this case $s'\sigma' \simeq t'\sigma'$ is not a tautology of the form $w \simeq w$ since $t'\sigma' \not\equiv t$, given that $t' \triangleright y$. Note that we have $t \overset{+}{\curvearrowright}_{\overline{\mathcal{S}}} t[x_1]_{p_1} \ldots [x_n]_{p_n} \cdot \{x_1 \mapsto r_1, \ldots, x_n \mapsto r_n\}$ for some variables $x_1, \ldots, x_n$, where $\forall i \in \{1 \ldots n\} : t|_{p_i} \to r_i \in S$ and $t[r_1]_{p_1} \ldots [r_n]_{p_n} \equiv t'\sigma'$. We denote $t[x_1]_{p_1} \ldots [x_n]_{p_n}$ by t_0 and $\{x_1 \mapsto r_1, \ldots, x_n \mapsto r_n\}$ by δ. And we choose s' as u, t' as v and σ' as φ. Then the required conditions are fulfiled:
   - $u\varphi \equiv t$.
   - $v\varphi \equiv t_0\delta$ with $t \overset{+}{\curvearrowright}_{\overline{\mathcal{S}}} t_0 \cdot \delta$.
   - $\forall q \in red\_pos(t_0 \cdot \delta, \curvearrowright_{\overline{\mathcal{S}}}) : q$ is a valid non-variable position in v. By definition of φ and $\curvearrowright_{\overline{\mathcal{S}}}$.

- $yellow\_vars(v, t_0) \cap (vars(u) \cup green\_vars(v, t_0)) = \emptyset$. On the one hand we have $yellow\_vars(v, t_0) \cap vars(u) = \emptyset$, since $u = y$ and, given that $y\sigma \equiv s \curvearrowright_{\overline{\mathcal{S}}} x \cdot \{x \mapsto t\}$ for some variable x, we have $y \notin yellow\_vars(v, t_0)$ by definition of φ. On the other hand, we have $yellow\_vars(v, t_0) \cap green\_vars(v, t_0) = \emptyset$ since, for any variable y in $vars(v)$ and positions q_1 and q_2 such that $v|_{q_1} \equiv y$ and $v|_{q_2} \equiv y$, we have $t_0|_{q_1} \equiv t_0|_{q_2}$ and, hence, y is in $yellow\_vars(v, t_0)$ or in $green\_vars(v, t_0)$, but not in both.

2. $s'\sigma' \equiv s$.
   (a) $s' = y \in \mathcal{X} \wedge y \notin vars(t')$. Then let $\varphi = \sigma' \cup \{y \mapsto t\}$. We choose s' as u and t' as v. The conditions are fulfiled for the same reasons as in the previous case, except for $yellow\_vars(v, t_0) \cap vars(u) = \emptyset$, which is now due to the fact that $y \notin vars(v)$.
   (b) $s' \notin \mathcal{X} \vee s' = y \in vars(t')$. Then there is an inference

$$\frac{s'' \simeq t'' \qquad s' \simeq t'}{(t'' \simeq t')\gamma}$$

where the left premise $s'' \simeq t''$ is a variable renamed version of the right premise $s' \simeq t'$ and $\gamma = mgu(s'', s')$. Note that this is a correct inference since, if s' is a variable y, then $y \in vars(t')$ and s'' is also a variable. Note also that since γ is simply a renaming substitution we can assume, wlog., that $s''\gamma \equiv s'$. Then $vars(t''\gamma) \subseteq vars(t'') \cup vars(s')$ and $t'\gamma \equiv t'$. Now, since $s'' \simeq t''$ is a variable renamed version of $s' \simeq t'$, we have that $vars(t'') \cap vars(t') = \emptyset$ and, hence, $vars(t'') \cap V = \emptyset$. And we also have $vars(s') \cap V = \emptyset$, since $s'\sigma' \equiv s$. Therefore $vars(t''\gamma) \cap V = \emptyset$, and hence and $t''\gamma \simeq t'$ has an instance of the form $t \simeq t'\sigma'$ under some substitution φ. This instance is clearly not a tautology of the form $w \simeq w$, and hence $t''\gamma \simeq t'$ is in E. We choose $t''\gamma$ as u and t' as v. As before, we have $t \overset{+}{\curvearrowright}_{\overline{\mathcal{S}}} t[x_1]_{p_1} \ldots [x_n]_{p_n} \cdot \{x_1 \mapsto r_1, \ldots, x_n \mapsto r_n\}$ for some variables $x_1, \ldots, x_n$, where $\forall i \in \{1 \ldots n\} : t|_{p_i} \rightarrow r_i \in S$ and $t[r_1]_{p_1} \ldots [r_n]_{p_n} \equiv t'\sigma'$. We denote $t[x_1]_{p_1} \ldots [x_n]_{p_n}$ by t_0 and $\{x_1 \mapsto r_1, \ldots, x_n \mapsto r_n\}$ by δ. Again, the required conditions are fulfiled for the same reasons as in case 1, except for $yellow\_vars(v, t_0) \cap vars(u) = \emptyset$, that is, $yellow\_vars(t', t_0) \cap vars(t''\gamma) = \emptyset$, which is now due to the fact that $yellow\_vars(t', t_0) \subseteq V$ and, as seen before, $vars(t''\gamma) \cap V = \emptyset$.

□

Lemma 19. *Let $u \simeq v$ be a equation in E and $\mathcal{S}$ a subset of $\vec{E}_{\mathcal{B}}$ with non-overlapped left-hand sides. If, for some substitution φ, we have*

1. *$v\varphi \equiv t_0\delta$ with $u\varphi \overset{+}{\curvearrowright}_{\overline{\mathcal{S}}} t_0 \cdot \delta$,*
2. *$\forall q \in red\_pos(t_0 \cdot \delta, \curvearrowright_{\overline{\mathcal{S}}}) : q$ is a valid non-variable position in v, and*
3. *$yellow\_vars(v, t_0) \cap (vars(u) \cup green\_vars(v, t_0)) = \emptyset$.*

then there is an equation $u' \simeq v'$ in E such that, for some ground substitution ψ, we have

1. *$u'\psi \equiv u\varphi$,*

2. $v'\psi \equiv t_1\rho$ *with* $t_0 \cdot \delta \overset{!}{\curvearrowright}_{\overline{\mathcal{S}}} t_1 \cdot \rho$ *and*
3. $yellow\_vars(v', t_1) \cap (vars(u') \cup green\_vars(v', t_1)) = \emptyset$.

Proof. We proceed by induction on $\overset{+}{\curvearrowright}_{\mathcal{S}}$. If $t_0 \cdot \delta$ is irreducible wrt. $\curvearrowright_{\overline{\mathcal{S}}}$ then we are done: we choose u as u', v as v', φ as ψ and $t_0 \cdot \delta$ as $t_1 \cdot \rho$. Otherwise, let p be a position in $red\_pos(t_0 \cdot \delta, \curvearrowright_{\overline{\mathcal{S}}})$. Then $t_0\delta|_p \equiv l$ for some rule $l \to r$ in $\mathcal{S}$, and there is some equation $l' \simeq r'$ in E such that (wlog.) $l'\varphi \equiv l$ and $r'\varphi \equiv r$. Then there is an inference

$$\frac{l' \simeq r' \qquad u \simeq v}{(u \simeq v[r']_p)\theta}$$

where $\theta = mgu(v|_p, l')$, whose conclusion has an instance of the form $u\varphi \simeq v\varphi[r]_p$ under some substitution ψ. Note that $v|_p$ is not a variable, by condition 2. Note also that we have $u\varphi \overset{+}{\curvearrowright}_{\overline{\mathcal{S}}} t_0 \cdot \delta \curvearrowright_{\overline{\mathcal{S}}} t_0[x]_p \cdot (\delta \cup \{x \mapsto r\})$ for some variable x. Let t'_0 denote $t_0[x]_p$ and δ' denote $\delta \cup \{x \mapsto r\}$. We have $v\varphi \equiv t_0\delta$, $v\varphi[r]_p \equiv t'_0\delta'$. Neither $u\varphi \simeq v\varphi$ nor $u\varphi \simeq v\varphi[r]_p$ are tautologies of the form $w \simeq w$, by Lemma 16. Now we show that the remaining conditions are preserved for the conclusion of the inference. We have $\forall q \in red\_pos(t'_0 \cdot \delta', \curvearrowright_{\overline{\mathcal{S}}}) : q$ is a valid non-variable position in $v[r']_p\theta$, since $v\theta[\,]_p$ is a particularization of $v[\,]_p$. Wrt. $yellow\_vars(v[r']_p\theta, t'_0) \cap (vars(u\theta) \cup green\_vars(v[r']_p\theta, t'_0)) = \emptyset$, note that we have $vars(t_0|_p) = \emptyset$ by definition of $\curvearrowright_{\overline{\mathcal{S}}}$ and, hence, $vars(v|_p) \subseteq green\_vars(v, t_0)$. Therefore $vars(v|_p) \cap yellow\_vars(v, t_0) = \emptyset$, since $yellow\_vars(v, t_0) \cap green\_vars(v, t_0) = \emptyset$. Moreover we have $vars(l') \cap yellow\_vars(v, t_0) = \emptyset$, since $vars(l') \cap vars(v) = \emptyset$. Then $(vars(v|_p) \cup vars(l')) \cap yellow\_vars(v, t_0) = \emptyset$ and, since $Dom(\theta) \subseteq vars(v|_p) \cup vars(l')$, we have $Dom(\theta) \cap yellow\_vars(v, t_0) = \emptyset$. This fact, together with $v|_p \notin \mathcal{X}$, gives us $yellow\_vars(v[r']_p\theta, t'_0) = yellow\_vars(v, t_0)$. Now note that we also have $vars(Im(\theta)) \subseteq vars(v|_p) \cup vars(l')$ and, hence, $vars(Im(\theta)) \cap yellow\_vars(v, t_0) = \emptyset$. Then, on the one hand, we have $yellow\_vars(v, t_0) \cap vars(u\theta) = \emptyset$, given that $yellow\_vars(v, t_0) \cap vars(u) = \emptyset$. On the other hand, we have $yellow\_vars(v, t_0) \cap green\_vars(v[r']_p\theta, t'_0) = \emptyset$, since $yellow\_vars(v, t_0) \cap green\_vars(v, t_0) = \emptyset$ and, moreover, $vars(r') \cap yellow\_vars(v, t_0) = \emptyset$, since $vars(r') \cap vars(v) = \emptyset$. Therefore we have $yellow\_vars(v, t_0) \cap (vars(u\theta) \cup green\_vars(v[r']_p\theta, t'_0)) = \emptyset$ and hence $yellow\_vars(v[r']_p\theta, t'_0) \cap (vars(u\theta) \cup green\_vars(v[r']_p\theta, t'_0)) = \emptyset$, as $yellow\_vars(v[r']_p\theta, t'_0) = yellow\_vars(v, t_0)$.

Once seen that the conditions are preserved, we conclude by induction hypothesis, since $t_0 \cdot \delta \curvearrowright_{\overline{\mathcal{S}}} t'_0 \cdot \delta'$ and $\curvearrowright_{\overline{\mathcal{S}}} \subseteq \curvearrowright_{\mathcal{S}}$. □

Lemma 20. *Let t be a term in $\mathcal{B}$. If t is self-returning for some rule $l \to r$ in $\vec{E}_{\mathcal{B}}$ then all positions in $red\_pos(arg\_abs(t, \mathcal{R}), \curvearrowright_{\vec{E}_{\mathcal{B}}})$ are overlapped and l occurs only once in t.*

Proof. We proceed by contradiction. Let $s \to t$ be the minimal rule wrt. $\succ_E$ in $\mathcal{R}$ with t as right-hand side. Since t is self-returning (see definition 16) for $l \to r$ we know that s is of the form $C[r]_p$ and t is of the form $C[l]_p$ for some position p not in $P_{\mathcal{B}^{\triangleleft t}}(t)$. Now assume there is some rule $l_1 \to r_1$ (possibly $l \to r$) in $\vec{E}_{\mathcal{B}}$ reducing t at some position p' (possibly p) not in $P_{\mathcal{B}^{\triangleleft t}}(t)$ which reduces t at some position p'' which is not overlapped with p. Then, by Lemma 9, some rule $l'_1 \to r'_1$ in $\mathcal{R}$ would reduce t at position p' or at some non-$\mathcal{B}$-position below p'. So given that $s \equiv t[r]_p$, we would have $s \rhd l_1$ and hence $s \to t \succ_E l'_1 \to r'_1$, contradicting Lemma 17. $\square$

Lemma 21. *Let t be a term in $\mathcal{B}$ and $s \to t$ the minimal rule wrt. $\succ_E$ in $\mathcal{R}$ with t as right-hand side. If $|s| < |t|$, s is of the form $C[r]_p$ and t is of the form $C[l]_p$ for some rule $l \to r$ in $\vec{E}_{\mathcal{B}}$ and position p not in $P_{\mathcal{B}^{\lhd t}}(t)$ then $C[x]_p \cdot \{x \mapsto r\} \overset{!}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} w \cdot \gamma$ with $w\gamma \equiv C[r]_p$.*

Proof. We assume Lemma 26 is true for terms whose size is smaller than the size of t. If $C[x]_p \cdot \{x \mapsto r\}$ is irreducible wrt. $\rightsquigarrow_{\vec{E}_{\mathcal{B}}}$ then we are done. Otherwise let D denote a rewrite sequence of the form $C[x]_p \cdot \{x \mapsto r\} \overset{!}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} w \cdot \gamma$. Then D is of the form $t_1 \cdot \alpha_1 \overset{\parallel}{\curvearrowright}_{\underline{\{l_1 \to r_1\}}} t_2 \cdot \alpha_2 \ldots t_{n-1} \cdot \alpha_{n-1} \overset{\parallel}{\curvearrowright}_{\underline{\{l_{n-1} \to r_{n-1}\}}} t_n \cdot \alpha_n$, where $n > 1$ and $\forall i \in 1 \ldots n-1$ we have $l_i \to r_i \in \vec{E}_{\mathcal{B}}$ and there is no rule $l'_i \to r'_i \in \vec{E}_{\mathcal{B}}$ with $l_i \rhd l'_i$ reducing $t_i \cdot \alpha_i$ wrt. $\curvearrowright_{\underline{\{l'_i \to r'_i\}}}$. Recall that, by Lemma 13, this rewrite sequence D is unique (up to permutation of non-overlapped steps). Moreover, since $C[x]_p$ has only one variable, then necessarily any rewrite subsequence $t_i \cdot \alpha_i \overset{\parallel}{\curvearrowright}_{\underline{\{l_i \to r_i\}}} t_{i+1} \cdot \alpha_{i+1}$ has only one step. Then the rewrite sequence D_E of the form $t_1 \cdot \alpha_1 \curvearrowright_{\{l_1 \to r_1\}} t_2 \cdot \alpha_2 \ldots t_{n-1} \cdot \alpha_{n-1} \curvearrowright_{\{l_{n-1} \to r_{n-1}\}} t_n \cdot \alpha_n$ is equivalent to D. Now we put our attention to the first (and possibly only) step in D_E. We have $t_1 \equiv C[x]_p$ and $t_2 \equiv C[y]_{p'}$ for some variable y and position p' above p. Note that $p' \not\equiv p$ since, by definition of $\curvearrowright_{\underline{\vec{E}_{\mathcal{B}}}}$, p' must be a non-variable position of t_1. We have that p' is a $\mathcal{B}$-position of s by Lemma 10 since $s \to t \in \mathcal{R}$, $l_1 \to r_1 \in \vec{E}_{\mathcal{B}}$, $s|_{p'} \equiv l_1$ and $p' \neq \lambda$ (by definition of $\rightsquigarrow_{\vec{E}_{\mathcal{B}}}$). Now let p'_b be the highest position of s above p' such that $s|_{p'_b}$ is a $\mathcal{B}$-term (this position exists since p' is a $\mathcal{B}$-position of s). Then $s|_{p'_b}$ is the biggest (in size) $\mathcal{B}$-term in s above p'. Let w denote $s|_{p'_b}$. We have $|w| < |t|$ since $|s| < |t|$. Moreover

- $p'_b \neq \lambda$ since otherwise we would have $s \in \mathcal{B}$ and $s \in Left(\mathcal{R})$, contradicting Lemma 6, and
- for any position q' below p'_b in s such that $q' \neq p'$ but $s|_{q'} \equiv l_1$ we have that $s|_{q'_b}$ is a $\mathcal{B}$-term for some position q'_b above q' which is not overlapped with p':
  Let q' be any position below p'_b in s such that $q' \neq p'$ but $s|_{q'} \equiv l_1$. Then q' is not overlapped with p' and hence $s|_{q'} \equiv t|_{q'}$, since $s \equiv t[l_1]_{p'}$. And we have $s \to t \succ_E l_1 \to r_1$ since $p' \neq \lambda$. Now recall that $l_1 \to r_1 \in \vec{E}_{\mathcal{B}}$. Then, by Lemma 5, we have that there is some rule $l'_1 \to r'_1$ (possibly $l_1 \to r_1$) in $\mathcal{R}$ which reduces l_1 at some non-$\mathcal{B}$-position. Moreover $l_1 \notin \mathcal{B}$ by Lemma 7. Then, given that $s \to t$ is the minimal rule wrt. $\succ_E$ in $\mathcal{R}$ with t as right-hand side, we must have $t|_{q'_b}$ is a $\mathcal{B}$-term for some position q'_b above q', since otherwise we would have $l'_1 \to r'_1 \succeq_E s \to t$ by Lemma 17, contradicting $s \to t \succ_E l_1 \to r_1$ since $l_1 \trianglerighteq l'_1$ implies $l_1 \to r_1 \succeq_E l'_1 \to r'_1$. This position q'_b is clearly not below p', since q' is not overlapped with p' and q'_b is above q'. But q'_b can neither be above p' since p, which is below p', is not in $P_{\mathcal{B}^{\lhd t}}(t)$.

Let $\mathcal{S}_w$ denote $\{l_1 \to r_1\}$ and D^w denote a rewrite sequence of the form $w \overset{!}{\curvearrowright}_{\overline{\mathcal{S}_w}} w_1 \cdot \beta_1 \overset{!}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} w_{n'} \cdot \beta_{n'}$. Since, as seen before, $l_1 \to r_1 \in \vec{E}_{\mathcal{B}}$ and $|w| < |t|$, we can apply Lemma 26 choosing w as t and $\mathcal{S}_w$ as $\mathcal{S}$, and we have $w_{n'}\beta_{n'} \equiv w$ or $w_{n'}\beta_{n'} \to w \in \vec{E}_{\mathcal{B}}$. Now consider the rewrite sequence $D^w(\curvearrowright_{\vec{E}_{\mathcal{B}}})$. Since $s \equiv C[w]_{p'_b}$, then the rewrite sequence D'_E consisting of the same steps of $D^w(\curvearrowright_{\vec{E}_{\mathcal{B}}})$ but where the context $C[\,]_{p'_b}$ is added to each term is a rewrite sequence with $\curvearrowright_{\vec{E}_{\mathcal{B}}}$ whose first term corresponds to s and whose last term corresponds to $C[w_{n'}]_{p'_b} \cdot \beta_{n'}$. If $w_{n'}\beta_{n'} \equiv w$ then $C[w_{n'}]_{p'_b} \cdot \beta_{n'}$ is irreducible wrt. $\rightsquigarrow_{\vec{E}_{\mathcal{B}}}$, since $w_{n'} \cdot \beta_{n'}$ is irreducible wrt. $\rightsquigarrow_{\vec{E}_{\mathcal{B}}}$, $w \notin Left(\vec{E}_{\mathcal{B}})$ by Lemma 7, and positions higher than p'_b are also irreducible wrt. $\rightsquigarrow_{\vec{E}_{\mathcal{B}}}$ by Lemma 10, given that p'_b is the highest position of s above p' such that $s|_{p'_b}$ is a $\mathcal{B}$-term (and hence higher positions are not $\mathcal{B}$-positions), $C[w]_{p'_b} \equiv s$ and $s \to t \in \mathcal{R}$. If $w_{n'}\beta_{n'} \to w \in \vec{E}_{\mathcal{B}}$ then we have not only $C[w_{n'}]_{p'_b} \cdot \beta_{n'} \curvearrowright_{\vec{E}_{\mathcal{B}}} C[z]_{p'_b} \cdot \{z \mapsto w\}$ but also $C[w_{n'}]_{p'_b} \cdot \beta_{n'} \rightsquigarrow_{\vec{E}_{\mathcal{B}}} C[z]_{p'_b} \cdot \{z \mapsto w\}$, since $p'_b \neq \lambda$. And $C[z]_{p'_b} \cdot \{z \mapsto w\}$ is irreducible wrt. $\rightsquigarrow_{\vec{E}_{\mathcal{B}}}$ by Lemma 10 for the same reason as before. Then in either case there is a rewrite sequence D''_E of the form $t'_1 \cdot \alpha'_1 \curvearrowright_{\{l'_1 \to r'_1\}} t'_2 \cdot \alpha'_2 \ldots t'_{m-1} \cdot \alpha'_{m-1} \curvearrowright_{\{l'_{m-1} \to r'_{m-1}\}} t'_m \cdot \alpha'_m$, with $t'_1 \cdot \alpha'_1 \equiv C[r]_p \cdot \{\}$, $t'_m\alpha'_m \equiv C[w]_{p'_b}$ and $t'_m \cdot \alpha'_m$ irreducible wrt. $\rightsquigarrow_{\vec{E}_{\mathcal{B}}}$.

Now let $i_1 \ldots i_k$ denote the set of indices i in $1 \ldots m$ such that the i-*th* step in D''_E is applied at a position above p (note that, although no steps are applied in D''_E at or below p, there may be some steps applied at positions which are not overlapped with p). We will prove that for any step of the form $t'_{i_j} \cdot \alpha'_{i_j} \curvearrowright_{\{l'_{i_j} \to r'_{i_j}\}} t'_{i_j}[x'_{i_j}]_{p'_{i_j}} \cdot \alpha'_{i_j+1}$ in D''_E there is a step of the form $t_j \cdot \alpha_j \curvearrowright_{\{l'_{i_j} \to r'_{i_j}\}} t_j[x_j]_{p'_{i_j}} \cdot \alpha_{j+1}$ in D_E, i.e. there is a one-to-one correspondence between the steps in D''_E and the first k steps in D_E. By induction on j:

– If $j = 1$ then no previous step to the i_j-*th* in D''_E has been applied at any position above p. Then, although possibly $i_j > 1$, we have $p'_{i_j} = p'$ and hence $l'_{i_j} \to r'_{i_j} = l_1 \to r_1$, since $t'_1 \cdot \alpha'_1 \equiv C[r]_p \cdot \{\}$ and $C[r]_p|_{p'} \equiv l_1$. And moreover we have $t_1 \cdot \alpha_1 \curvearrowright_{\{l_1 \to r_1\}} t_1[x_1]_{p'} \cdot \{x_1 \mapsto r_1\}$ in D_E, since $t_1 \cdot \alpha_1 \equiv C[x]_p \cdot \{x \mapsto r\}$.
– Now we prove the induction step, namely that if $t_j \cdot \alpha_j \curvearrowright_{\{l'_{i_j} \to r'_{i_j}\}} t_j[x_j]_{p'_{i_j}} \cdot \alpha_{j+1}$ and $j < k$ then $t_{j+1} \cdot \alpha_{j+1} \curvearrowright_{\{l'_{i_{j+1}} \to r'_{i_{j+1}}\}} t_{j+1}[x_{j+1}]_{p'_{i_{j+1}}} \cdot \alpha_{j+2}$. For this, consider the i_{j+1}-*th* step in D''_E. It is of the form $t'_{i_{j+1}} \cdot \alpha'_{i_{j+1}} \curvearrowright_{\{l'_{i_{j+1}} \to r'_{i_{j+1}}\}} t'_{i_{j+1}}[x'_{i_{j+1}}]_{p'_{i_{j+1}}} \cdot \alpha'_{i_{j+1}+1}$. We can prove that
  - $t'_{i_{j+1}}\alpha'_{i_{j+1}}|_{p'_{i_j}} \equiv r'_{i_j}$ and $t_{j+1}\alpha_{j+1}|_{p'_{i_j}} \equiv r'_{i_j}$: Wrt. $t'_{i_{j+1}}\alpha'_{i_{j+1}}|_{p'_{i_j}} \equiv r'_{i_j}$, recall that the i_{j+1}-*th* step in D''_E is the first one after the i_j-*th* which is applied at a position above p'_{i_j}. Wrt. $t_{j+1}\alpha_{j+1}|_{p'_{i_j}} \equiv r'_{i_j}$, recall that $t_{j+1} \cdot \alpha_{j+1} \equiv t_j[x_j]_{p'_{i_j}} \cdot \{x_j \mapsto r'_{i_j}\}$.
  - $t'_{i_{j+1}}\alpha'_{i_{j+1}}[\,]_{p'_{i_j}}|_{p'_{i_{j+1}}} \equiv t_{j+1}\alpha_{j+1}[\,]_{p'_{i_j}}|_{p'_{i_{j+1}}}$: By contradiction. Let us suppose that there is some position q'' below $p'_{i_{j+1}}$ which is not overlapped with p'_{i_j} and such that $t'_{i_{j+1}}\alpha'_{i_{j+1}}|_{q''} \not\equiv t_{j+1}\alpha_{j+1}|_{q''}$. Then q'' must be a prefix of some position q' in s such that $s|_{q'} \equiv l_1$ (otherwise we

would have $t'_{i_{j+1}}\alpha'_{i_{j+1}}|_{q''} \equiv t'_1\alpha'_1|_{q''} \equiv C[r]_p|_{q''} \equiv t_1\alpha_1|_{q''}$, and $t_1\alpha_1|_{q''} \equiv t_{j+1}\alpha_{j+1}|_{q''}$ since q'' is not overlapped with p'_{i_j}). In this case, as stated before, we have that $s|_{q'_b}$ is a $\mathcal{B}$-term for some position q'_b above q' which is not overlapped with p', and hence neither with p'_{i_j}. Therefore we have $w \equiv s|_{p'_b} \rhd s|_{q'_b} \equiv w'$. Wlog. we assume that q'_b is the highest $\mathcal{B}$-position above q' in s which is not overlapped with p. Now note that the innermost rewriting strategy of the rewrite sequence D^w on w implies an innermost rewriting strategy on its subterm w'. And, by Lemma 26, we have that if $w' \overset{!}{\curvearrowright}_{\mathcal{S}_w} w'_1 \cdot \rho' \overset{!}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} w'_{m'} \cdot \beta'$ then $w'_{m'}\beta' \equiv w'$ or $w'_{m'}\beta' \to w' \in \vec{E}_{\mathcal{B}}$. Therefore some term of the form $t'_{k'} \cdot \alpha'_{k'}$ with $t'_{k'}\alpha'_{k'}|_{q'_b} \equiv w'$ is obtained in D''_E before the i_{j+1}-*th* step (observe that $p'_{i_{j+1}}$ is above q'_b since $p'_{i_{j+1}}$ is overlapped with p', q'_b is not overlapped with p', q'_b is above q' and q' is below $p'_{i_{j+1}}$; therefore in the case that $w'_{m'}\beta' \not\equiv w'$ then the rule $w'_{m'}\beta' \to w' \in \vec{E}_{\mathcal{B}}$ would be applied before the i_{j+1}-*th* step given that, by Lemma 8, there are no two different rules in $\vec{E}_{\mathcal{B}}$ with the same left-hand side). Now recall that $t_{j+1}\alpha_{j+1}|_{q''} \equiv C[r]_p|_{q''}$ since q'' is not overlapped with p'_{i_j}. Therefore, since the i_{j+1}-*th* step in D''_E is the first one after the i_j-*th* which is applied at a position above p'_{i_j}, if $t'_{i_{j+1}}\alpha'_{i_{j+1}}|_{q''} \not\equiv t_{j+1}\alpha_{j+1}|_{q''}$ then necessarily some step in D''_E before the i_{j+1}-*th* must have been applied at some position above q'_b which is not overlapped with p'_{i_j} and hence neither with p. Let us consider the first step in D''_E which is applied at some position q''_b above q'_b which is not overlapped with p is the k''-*th*. Then we have $t'_{k''}\alpha'_{k''}|_{q''_b} \equiv C[r]_p|_{q''_b} \equiv s|_{q''_b}$. Moreover $q''_b \neq \lambda$, since q''_b is not overlapped with p. Therefore there is a rule in $\vec{E}_{\mathcal{B}}$ whose left-hand side is $s|_{q''_b}$ and $s \rhd s|_{q''_b}$. And, since t is of the form $C[l]_p$, we have $t|_{q''_b} \equiv s|_{q''_b}$. Now observe that q'_b is the highest $\mathcal{B}$-position above q' in $P_{\mathcal{B}^{\lhd t}}(t)$, since p is not in $P_{\mathcal{B}^{\lhd t}}(t)$ and q'_b is the highest $\mathcal{B}$-position above q' in s which is not overlapped with p. Then q''_b is a position not in $P_{\mathcal{B}^{\lhd t}}(t)$. Moreover $t|_{q''_b}$ is reducible by $\mathcal{R}$ at some non-$\mathcal{B}$-position by Lemma 5, and hence t is reducible by some rule $l'' \to r''$ in $\mathcal{R}$ at some position not in $P_{\mathcal{B}^{\lhd t}}(t)$. But then, by Lemma 17, we have $l'' \to r'' \succeq_E s \to t$, which is a contradiction with $s \rhd s|_{q''_b} \equiv t|_{q''_b} \unrhd l''$.

Therefore we have $t'_{i_{j+1}}\alpha'_{i_{j+1}}|_{p'_{i_{j+1}}} \equiv t_{j+1}\alpha_{j+1}|_{p'_{i_{j+1}}}$. Finally recall that, by definition of $\rightsquigarrow_{\vec{E}_{\mathcal{B}}}$, there is no rule $l' \to r'$ in $\vec{E}_{\mathcal{B}}$ with $l'_{i_{j+1}} \rhd l'$ reducing $t'_{i_{j+1}}\alpha'_{i_{j+1}}$ at a position above p'_{i_j}. And there is neither no rule $l'' \to r''$ in $\vec{E}_{\mathcal{B}}$ with $l_{j+1} \rhd l''$ reducing $t_{j+1}\alpha_{j+1}$ at a position above p'_{i_j}. Then, by Lemma 8, we have $l_{j+1} \to r_{j+1} = l'_{i_{j+1}} \to r'_{i_{j+1}}$ and hence $t_{j+1} \cdot \alpha_{j+1} \curvearrowright_{\{l'_{i_{j+1}} \to r'_{i_{j+1}}\}} t_{j+1}[x_{j+1}]_{p'_{i_{j+1}}} \cdot \alpha_{j+2}$.

Moreover we have

- $t'_m\alpha'_m|_{p'_{i_k}} \equiv r'_{i_k}$ since no steps after the i_k-*th* in D''_E are applied at a position above p, and

– $t'_m\alpha'_m[\,]_{p'_{i_k}}|_{p'_b} \equiv t_{k+1}\alpha_{k+1}[\,]_{p'_{i_k}}|_{p'_b}$, with a reasoning identical to the one used before to prove that $t'_{i_{j+1}}\alpha'_{i_{j+1}}[\,]_{p'_{i_j}}|_{p'_{i_{j+1}}} \equiv t_{j+1}\alpha_{j+1}[\,]_{p'_{i_j}}|_{p'_{i_{j+1}}}$, where m plays the role of i_{j+1}, k the role of j and p'_b the role of $p'_{i_{j+1}}$.

Therefore $t'_m\alpha'_m|_{p'_b} \equiv t_{k+1}\alpha_{k+1}|_{p'_b}$, since $t_{k+1} \cdot \alpha_{k+1} \equiv t_k[x_k]_{p'_{i_k}} \cdot \{x_k \mapsto r'_{i_k}\}$. Moreover $t'_m\alpha'_m[\,]_{p'_b} \equiv C[\,]_{p'_b}$ and $t'_{k+1}\alpha'_{k+1}[\,]_{p'_b} \equiv C[\,]_{p'_b}$, since there are no steps at positions above p'_b neither in D''_E (by definition) nor in D_E (since, as we have seen, the first k steps in D_E correspond to the steps in D''_E). So we have $t'_m\alpha'_m \equiv t_{k+1}\alpha_{k+1}$. Now recall that $t'_m \cdot \alpha'_m$ is irreducible wrt. $\rightsquigarrow_{\vec{E}_\mathcal{B}}$. Then also must be $t_{k+1} \cdot \alpha_{k+1}$, since its only blocked position is p'_{i_k}, that is, $P_\mathcal{X}(t_{k+1}) = \{p'_{i_k}\}$, and p'_{i_k} is also a variable position of $t'_m \cdot \alpha'_m$, since p'_{i_k} was blocked in the i_k*-th* step of D''_E, which is the last step applied at a position above p. Finally, the irreducibility of $t_{k+1} \cdot \alpha_{k+1}$ wrt. $\rightsquigarrow_{\vec{E}_\mathcal{B}}$ gives us $k+1 \equiv n$ and, since $t_{k+1}\alpha_{k+1} \equiv t'_m\alpha'_m \equiv C[w]_{p'_b} \equiv s \equiv C[r]_p$, we are done. □

Lemma 22. *Let t be a term in $\mathcal{B}$. Let $\mathcal{S}$ be a non-empty subset of $\vec{E}_\mathcal{B}$ with non-overlapped left-hand sides and such that each rule in $\mathcal{S}$ reduces t at some position not in $P_{\mathcal{B}^{\triangleleft t}}(t)$. Then there is an equation $u \simeq v \in E$ such that, for some substitution σ, we have*

1. *$u\sigma \equiv t$,*
2. *$v\sigma \equiv t_n\alpha$ with $t \overset{!}{\curvearrowright}_{\overline{\mathcal{S}}} t_1 \cdot \rho \overset{*}{\rightsquigarrow}_{\vec{E}_\mathcal{B}} t_n \cdot \alpha$, and*
   (a) $t_n \cdot \alpha \equiv t_1 \cdot \rho!_{\rightsquigarrow_{\vec{E}_\mathcal{B}}}$ and $t_n\alpha \to t \in \mathcal{R}$, or
   (b) $red\_vars(v, t_n \cdot \alpha, \curvearrowright_{\underline{\vec{E}_\mathcal{B}}}) \cap (vars(u) \cup green\_vars(v, t_n)) = \emptyset$.

Proof. First of all we must point out that, since rules in $\mathcal{S}$ have non-overlapped left-hand sides, then t has one and only one normal form wrt. $\curvearrowright_{\overline{\mathcal{S}}}$ by Lemma 12. And the rewrite sequence $t_1 \cdot \rho \overset{*}{\rightsquigarrow}_{\vec{E}_\mathcal{B}} t_n \cdot \alpha$ is unique (up to permutation of non-overlapped steps) by Lemma 13.

If t is non-self-returning for any rule in $\mathcal{S}$ then, by Lemma 18, we have there is an equation $u \simeq v \in E$ such that, for some substitution φ, we have

1. $u\varphi \equiv t$,
2. $v\varphi \equiv t_0\delta$ with $t \overset{+}{\curvearrowright}_{\overline{\mathcal{S}}} t_0 \cdot \delta$,
3. $\forall q \in red\_pos(t_0 \cdot \delta, \curvearrowright_{\overline{\mathcal{S}}}) : q$ is a valid non-variable position in v and
4. $yellow\_vars(v, t_0) \cap (vars(u) \cup green\_vars(v, t_0)) = \emptyset$.

Then, by Lemma 19, we have there is an equation $u' \simeq v'$ in E such that, for some ground substitution ψ, we have

1. $u'\psi \equiv u\varphi$,
2. $v'\psi \equiv t_1\rho$ with $t_0 \cdot \delta \overset{!}{\curvearrowright}_{\overline{\mathcal{S}}} t_1 \cdot \rho$ and
3. $yellow\_vars(v', t_1) \cap (vars(u') \cup green\_vars(v', t_1)) = \emptyset$.

Then we choose u' as u, v' as v, ψ as σ and $t_1 \cdot \rho$ as $t_n \cdot \alpha$ and we are done, since $red\_vars(v, t_1 \cdot \rho, \curvearrowright_{\underline{\vec{E}_{\mathcal{B}}}}) \subseteq yellow\_vars(v, t_1)$.

If, otherwise, t is self-returning (see definition 16) for some rule in $\mathcal{S}$ then, by Lemma 20, $\mathcal{S}$ must have one single rule $l \to r$ which reduces t at one single position p which is not in $P_{\mathcal{B}^{\triangleleft t}}(t)$. Now let $s' \simeq t'$ be an equation in E such that $s'\varphi \equiv s$ and $t'\varphi \equiv t$ for some substitution φ, where $s \to t$ is the minimal rule in $\mathcal{R}$ wrt. $\succ_E$ with t as right-hand side. Note that we have $s \equiv t[r]_p$ and p is a non-variable position of t', since t is self-returning for $l \to r$. Let t_1 denote $t[x]_p$ and ρ denote $\{x \mapsto r\}$. We clearly have $t_1 \cdot \rho \equiv t!_{\curvearrowright_{\underline{\mathcal{S}}}}$ since $l \to r$ reduces t only at position p. The following cases have to be considered:

1. $t_1 \cdot \rho$ is irreducible wrt. $\rightsquigarrow_{\vec{E}_{\mathcal{B}}}$. Then we choose t' as u, s' as v, φ as σ and $t_1 \cdot \rho$ as $t_n \cdot \alpha$, and we have
   - $u\sigma \equiv t$ since $t'\varphi \equiv t$,
   - $v\sigma \equiv t_n\alpha$ with $t \overset{!}{\curvearrowright}_{\underline{\mathcal{S}}} t_1 \cdot \rho \overset{*}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} t_n \cdot \alpha$ since, as seen before, we have $t \overset{!}{\curvearrowright}_{\underline{\mathcal{S}}} t_1 \cdot \rho$ and $v\sigma \equiv s'\varphi \equiv s \equiv t[r]_p \equiv t_1\rho \equiv t_n\alpha$, and
   - $t_n \cdot \alpha \equiv t_1 \cdot \rho!_{\rightsquigarrow_{\vec{E}_{\mathcal{B}}}}$ since $t_n \cdot \alpha \equiv t_1 \cdot \rho$ and $t_1 \cdot \rho$ is irreducible wrt. $\rightsquigarrow_{\vec{E}_{\mathcal{B}}}$, and $t_n\alpha \to t \in \mathcal{R}$ since $t_n\alpha \equiv v\sigma \equiv s'\sigma \equiv s$ and $s \to t \in \mathcal{R}$.
2. $t_1 \cdot \rho$ is reducible wrt. $\rightsquigarrow_{\vec{E}_{\mathcal{B}}}$. In this case we have $t_1 \cdot \rho \rightsquigarrow_{\vec{E}_{\mathcal{B}}} t'_1 \cdot \rho'$ for some abstracted term $t'_1 \cdot \rho'$. Then $t_1 \cdot \rho \overset{\parallel}{\curvearrowright}_{\underline{\{l_1 \to r_1\}}} t'_1 \cdot \rho'$ for some rule $l_1 \to r_1$ in $\vec{E}_{\mathcal{B}}$. Note that, since t_1 has only one variable, then necessarily the rewrite sequence $t_1 \cdot \rho \overset{\parallel}{\curvearrowright}_{\underline{\{l_1 \to r_1\}}} t'_1 \cdot \rho'$ has only one step, that is, we have $red\_pos(t_1 \cdot \rho, \rightsquigarrow_{\vec{E}_{\mathcal{B}}}) = \{p'\}$ for some prefix p' of p and $t_1 \cdot \rho \equiv t[x]_p \cdot \{x \mapsto r\} \curvearrowright_{\underline{\{l_1 \to r_1\}}} t[y]_{p'} \cdot \{y \mapsto r_1\} \equiv t'_1 \cdot \rho'$. Moreover $p' \not\equiv p$ since, by definition of $\curvearrowright_{\underline{\vec{E}_{\mathcal{B}}}}$, p' must be a non-variable position of t_1.
   (a) p' is a valid non-variable position of s'. In this case we choose t' as u, s' as v, φ as σ and $t_1 \cdot \rho$ as $t_n \cdot \alpha$ as in case 1. Conditions 1 and 2 are fulfiled as in case 1. But in this case condition 2b is fulfiled instead of condition 2a: we have $red\_vars(v, t_n \cdot \alpha, \curvearrowright_{\underline{\vec{E}_{\mathcal{B}}}}) = \emptyset$ since $t_n \cdot \alpha$ is reducible by $\curvearrowright_{\underline{\vec{E}_{\mathcal{B}}}}$ only at positions above p', and p' is a valid non-variable position of s'.
   (b) $s'|_{p''}$ is a variable z for some prefix p'' of p'.
      i. $t|_{p'} \equiv l_1$. This is not possible, since $s \equiv t[l_1]_{p'}$ and then we would have $s \equiv t$, contradicting $s \succ t$.
      ii. $t|_{p'} \rhd l_1$. In this case we have $|s| < |t|$ since $s \equiv t[l_1]_{p'}$. Now recall that $s \equiv t[r]_p$ and $t|_p \equiv l$, that is, s is of the form $C[r]_p$ and t is of the form $C[l]_p$. Then, by Lemma 21, we have $C[x]_p \cdot \{x \mapsto r\} \overset{!}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} w \cdot \gamma$ with $w\gamma \equiv C[r]_p$. Note that $w \cdot \gamma \equiv t_1 \cdot \rho!_{\rightsquigarrow_{\vec{E}_{\mathcal{B}}}}$, since t_1 denotes $t[x]_p$ and ρ denotes $\{x \mapsto r\}$. And moreover $w\gamma \equiv s$. Then we choose t' as u, s' as v and $w \cdot \gamma$ as $t_n \cdot \alpha$, and conditions 1, 2 and 2a are fulfiled.
      iii. $t|_{p'} \not\trianglerighteq l_1$. Since t is self-returning for $l \to r$ we have that p, and hence p', is a valid non-variable position of t'. Moreover since $s \equiv t[l_1]_{p'}$ and p'' is a prefix of p', we have $z\varphi \trianglerighteq l_1$. Therefore in this case we must have $t'|_{p'} \not\trianglerighteq z$.

A. s' is a variable. Then necessarily $s' = z$. In this case $z \notin vars(t')$ since, on the one hand, we have $t'|_{p'} \not\trianglerighteq z$ and hence $t'|_p \not\trianglerighteq z$ and, on the other hand, $t'|_q \not\equiv z$ for any position q which is not ovelapped with p, since otherwise we would have $s \not\equiv t[r]_p$, contradicting t is self-returning for $l \to r$. Then we choose t' as u, s' as v, φ as σ and $t_1 \cdot \rho$ as $t_n \cdot \alpha$ as in case 1. Conditions 1 and 2 are fulfiled for the same reasons. And we have $red\_vars(v, t_n \cdot \alpha, \curvearrowright_{\underline{\vec{E}_{\mathcal{B}}}}) \cap (vars(u) \cup green\_vars(v, t_n)) = \emptyset$, i.e. condition 2b is fulfiled: on the one hand we have $z \notin vars(u)$ since $z \notin vars(t')$ as we have seen; on the other hand, we have $green\_vars(v, t_n) = \emptyset$ since $v|_\lambda \equiv z$ and λ is above p (we have $p \neq \lambda$ since $t \notin Left(\mathcal{R})$ by Lemma 6) which is a variable position of t_n.

B. s' is not a variable. Then there is an inference

$$\frac{l' \simeq r' \qquad s' \simeq t'}{(s' \simeq t'[r']_p)\theta}$$

where $\theta = mgu(t'|_p, l')$ and $l' \simeq r'$ is an equation in E such that (wlog.) $l'\varphi \equiv l$ and $r'\varphi \equiv r$. Note that, although $(s'\theta \simeq t'[r']_p\theta)\varphi$ is a tautology of the form $w \simeq w$, since $s'\theta\varphi \equiv s \equiv t[r] \equiv t'[r']_p\theta\varphi$, we can prove there is another substitution φ' such that $(s'\theta \simeq t'[r']_p\theta)\varphi'$ is not a tautology: let φ' be a substitution such that $z\varphi' \equiv z\varphi[l]_p$ and $z'\varphi' \equiv z'\varphi$ for all other variables z; then we have $s'\theta\varphi' \equiv s[l]_p \equiv t[l]_p$ and $t'[r']_p\theta\varphi' \equiv t[r]_p$ since $s'|_{p''} \equiv z$ for some prefix p'' (of p' and) of p and $t'|_{p'} \not\trianglerighteq z$. Therefore $(s'\theta \simeq t'[r']_p\theta)\varphi'$ is not a tautology and hence we can assume $s'\theta \simeq t'[r']_p\theta$ is in E, and moreover $t'[r']_p\theta\varphi'|_{p'} \equiv l_1$. Now let $l_1' \simeq r_1'$ be an equation in E such that (wlog.) $l_1'\varphi \equiv l_1$ and $r_1'\varphi \equiv r_1$. Then there is an inference

$$\frac{l_1' \simeq r_1' \qquad s'\theta \simeq t'[r']_p\theta}{(s'\theta \simeq t'\theta[r_1']_{p'})\theta'}$$

where $\theta' = mgu(t'[r']_p\theta|_{p'}, l_1')$. Note that this is a correct inference since, although $p' \in P_{\mathcal{X}}(s', \varphi)$, p' is a non-variable position of $t'[r']_p\theta$ since it already was a non-variable position of t'. Moreover, although $s'\theta\theta'\varphi' \equiv s[l]_p$ we have $s'\theta\theta'\varphi \equiv s \equiv t[l_1]_{p'}$ and $t'\theta[r_1']_{p'}\theta'\varphi \equiv t[r_1]_{p'}$, since $t'|_{p'} \not\trianglerighteq z$ implies $z \notin Dom(\theta)$ and $z \notin Dom(\theta')$ (given that z does not occur neither in $l' \simeq r'$ nor in $l_1' \simeq r_1'$). Therefore $(s'\theta\theta' \simeq t'\theta[r_1']_{p'}\theta')\varphi$ is not a tautology and hence we can assume $s'\theta\theta' \simeq t'\theta[r_1']_{p'}\theta'$ is in E. Moreover, since s' is not a variable $s'\theta\theta'$ is not a variable either. Then there is an inference

$$\frac{s' \simeq t' \qquad s'\theta\theta' \simeq t'\theta[r_1']_{p'}\theta'}{(t' \simeq t'\theta[r_1']_{p'}\theta')\theta''}$$

where $\theta'' = mgu(s'\theta\theta', s')$, whose conclusion has an instance of the form $t \simeq t[r_1]_{p'}$ under the substitution φ. Now recall that we have $t \overset{!}{\curvearrowright}_{\overline{\mathcal{S}}} t_1 \cdot \rho \rightsquigarrow_{\vec{E}_\mathcal{B}} t_1' \cdot \rho'$ with $t_1'\rho' \equiv t[r_1]_{p'}$. Then $t \not\equiv t[r_1]_{p'}$ by Lemma 11, since $\curvearrowright_{\overline{\mathcal{S}}} \subseteq \curvearrowright_{\vec{E}_\mathcal{B}}$ and $\rightsquigarrow_{\vec{E}_\mathcal{B}} \subseteq \curvearrowright_{\vec{E}_\mathcal{B}}$, and $p' \neq \lambda$ by definition of $\rightsquigarrow_{\vec{E}_\mathcal{B}}$. Therefore we can assume $(t' \simeq t'\theta[r_1']_{p'}\theta')\theta''$ is in E. Then we choose $t'\theta''$ as u, $t'\theta[r_1']_{p'}\theta'\theta''$ as v, φ as σ and $t_1' \cdot \rho'$ as $t_n \cdot \alpha$. Conditions 1 and 2 are clearly fulfiled, and also condition 2b: note that $red\_vars(v, t_n \cdot \alpha, \curvearrowright_{\underline{\vec{E}_\mathcal{B}}}) = \emptyset$, since p' is the only variable position of t_n and there is no variable above p' in $t'\theta[r_1']_{p'}\theta'\theta''$, given that p' was a non-variable position of $t'[r']_p\theta$.

□

Lemma 23. *Let $t_1 \cdot \beta$ be an abstracted term wrt. $\vec{E}_\mathcal{B}$. Let $l \to r$ be a rule in $\vec{E}_\mathcal{B}$ with $l \not\equiv t_1\beta$, and such that there is no rule $s \to t$ in $\vec{E}_\mathcal{B}$ with $l \rhd s$ reducing $t_1 \cdot \beta$ wrt. $\curvearrowright_{\underline{\{s \to t\}}}$. If there is an equation $u \simeq v$ in E such that, for some substitution σ, we have*

1. *$u\sigma \curvearrowright_{\vec{E}_\mathcal{B}} u\sigma[x]_{p_0} \cdot \varepsilon \overset{*}{\curvearrowright}_{\vec{E}_\mathcal{B}} t_0 \cdot \alpha$ for some position p_0 not in $P_{\mathcal{B}^{\lhd u\sigma}}(u\sigma)$,*
2. *$v\sigma \equiv t_2\gamma$ with $t_0 \cdot \alpha \overset{*}{\curvearrowright}_{\vec{E}_\mathcal{B}} t_1 \cdot \beta \overset{*}{\curvearrowright}_{\underline{\{l \to r\}}} t_2 \cdot \gamma$,*
3. *$\forall q \in red\_pos(t_1 \cdot \beta, \curvearrowright_{\underline{\{l \to r\}}}) : t_2|_q \in \mathcal{X}$ or $t_2|_q \equiv t_1|_q$, and*
4. *$red\_vars(v, t_2 \cdot \gamma, \curvearrowright_{\underline{\vec{E}_\mathcal{B}}}) \cap (vars(u) \cup green\_vars(v, t_2)) = \emptyset$,*

then there is an equation $u' \simeq v'$ in E such that, for some substitution σ', we have

1. *$u'\sigma' \equiv u\sigma$,*
2. *$v'\sigma' \equiv t_3\delta$ with $t_1 \cdot \beta \overset{*}{\curvearrowright}_{\underline{\{l \to r\}}} t_3 \cdot \delta$,*
3. *$\forall q \in red\_pos(t_1 \cdot \beta, \curvearrowright_{\underline{\{l \to r\}}}) : t_3|_q \in \mathcal{X}$, and*
4. *$red\_vars(v', t_3 \cdot \delta, \curvearrowright_{\underline{\vec{E}_\mathcal{B}}}) \cap (vars(u') \cup green\_vars(v', t_3)) = \emptyset$.*

Proof. Let $Q = \{q \in red\_pos(t_1 \cdot \beta, \curvearrowright_{\underline{\{l \to r\}}}) \mid t_2|_q \equiv t_1|_q\}$. We proceed by induction on $\#Q$. If $\#Q = 0$ then we are done: we choose u as u', v as v', σ as σ', and $t_2 \cdot \gamma$ as $t_3 \cdot \delta$. Otherwise, let p be a position in Q. Now, from $t_1 \cdot \beta \overset{*}{\curvearrowright}_{\underline{\{l \to r\}}} t_2 \cdot \gamma$ and $p \in Q$ we have $p \in red\_pos(t_2 \cdot \gamma, \curvearrowright_{\underline{\{l \to r\}}})$. Moreover, note that $p \neq \lambda$, since $p \in red\_pos(t_1 \cdot \beta, \curvearrowright_{\underline{\{l \to r\}}})$ and $l \not\equiv t_1\beta$.

Two cases have to be considered:

1. $p \notin P_\mathcal{X}(v, \sigma)$. In this case we have that $u\sigma \simeq v\sigma$ is not a tautology of the form $w \simeq w$ by Lemma 11, since $t_2 \notin \mathcal{X}$ (since $t_2 \cdot \gamma$ is reducible wrt. $\curvearrowright_{\underline{\{l \to r\}}}$) and $\curvearrowright_{\underline{\{l \to r\}}} \subseteq \curvearrowright_{\vec{E}_\mathcal{B}}$. Therefore there is an inference

$$\frac{l' \simeq r' \qquad u \simeq v}{(u \simeq v[r']_p)\theta}$$

where $l' \simeq r'$ is an equation in E such that (wlog.) $l'\sigma \equiv l$ and $r'\sigma \equiv r$, where $\theta = mgu(v|_p, l')$, and whose conclusion has an instance of the form $u\sigma \simeq v\sigma[r]_p$ under some substitution σ'. Moreover we have $t_2 \cdot \gamma \curvearrowright_{\underline{\{l \to r\}}}$

$t_2[y]_p \cdot (\gamma \cup \{y \mapsto r\})$ for some variable y. Then we have that $u\sigma \simeq v\sigma[r]_p$ is not a tautology of the form $w \simeq w$ by Lemma 11, since $p \neq \lambda$. Therefore $u\theta \simeq v[r']_p\theta$ is in E, and it clearly fulfils the two first conditions required to $u \simeq v$. In order to prove the other required conditions to $u \simeq v$ are also fulfiled by $u\theta \simeq v[r']_p\theta$, let t'_2 denote $t_2[y]_p$ and γ' denote $\gamma \cup \{y \mapsto r\}$. We have $t'_2|_q \in \mathcal{X}$ or $t'_2|_q \equiv t_1|_q$ for all $q \in red\_pos(t_1 \cdot \beta, \curvearrowright_{\underline{\{l \to r\}}})$, since $t_2|_q \in \mathcal{X}$ or $t_2|_q \equiv t_1|_q$ for all $q \in red\_pos(t_1 \cdot \beta, \curvearrowright_{\underline{\{l \to r\}}})$, and $p \in red\_pos(t_1 \cdot \beta, \curvearrowright_{\underline{\{l \to r\}}})$. In order to prove $red\_vars(v[r']_p\theta, t'_2 \cdot \gamma', \curvearrowright_{\underline{\vec{E}_\mathcal{B}}}) \cap (vars(u\theta) \cup green\_vars(v[r']_p\theta, t'_2)) = \emptyset$ two properties will be useful. The first one being that, since $p \notin P_\mathcal{X}(v, \sigma)$, then for any variable z in v, if $z\theta$ is still a variable and it belongs to $red\_vars(v[r']_p\theta, t'_2 \cdot \gamma', \curvearrowright_{\underline{\vec{E}_\mathcal{B}}})$ then z already belonged to $red\_vars(v, t_2 \cdot \gamma, \curvearrowright_{\underline{\vec{E}_\mathcal{B}}})$. And the second one being that, for any variable z in $vars(v|_p)$, we have $z\theta$ not in $red\_vars(v[r']_p\theta, t'_2 \cdot \gamma', \curvearrowright_{\underline{\vec{E}_\mathcal{B}}})$: note that otherwise, by the first property, we would have z in $red\_vars(v, t_2 \cdot \gamma, \curvearrowright_{\underline{\vec{E}_\mathcal{B}}})$, and this would contradict there is no rule $s \to t$ in $\vec{E}_\mathcal{B}$ with $l \rhd s$ reducing $t_1 \cdot \beta$ wrt. $\curvearrowright_{\underline{\{s \to t\}}}$, given that $p \notin P_\mathcal{X}(v, \sigma)$. Now let z' be a variable in $vars(u\theta) \cup green\_vars(v[r']_p\theta, t'_2)$. The following cases have to be considered:

(a) $z' \in vars(u\theta)$. Then we have $z' \equiv z\theta$ for some variable z in $vars(u)$.
   i. $z \notin vars(v|_p)$. Then clearly $z\theta \equiv z$. In this case we have $z\theta$ not in $red\_vars(v[r']_p\theta, t'_2 \cdot \gamma', \curvearrowright_{\underline{\vec{E}_\mathcal{B}}})$ since otherwise, by the first property, we would have z in $red\_vars(v, t_2 \cdot \gamma, \curvearrowright_{\underline{\vec{E}_\mathcal{B}}})$, thus contradicting $red\_vars(v, t_2 \cdot \gamma, \curvearrowright_{\underline{\vec{E}_\mathcal{B}}}) \cap vars(u) = \emptyset$.
   ii. $z \in vars(v|_p)$. Then $z\theta$ not in $red\_vars(v[r']_p\theta, t'_2 \cdot \gamma', \curvearrowright_{\underline{\vec{E}_\mathcal{B}}})$ by the second property.

   Therefore $red\_vars(v[r']_p\theta, t'_2 \cdot \gamma', \curvearrowright_{\underline{\vec{E}_\mathcal{B}}}) \cap vars(u\theta) = \emptyset$.

(b) $z' \in green\_vars(v[r']_p\theta, t'_2)$.
   i. z' occurs in $v[\,]_p\theta$. Then we have $z' \equiv z\theta$ for some variable z in $green\_vars(v, t_2)$, and we can conclude $z\theta$ not in $red\_vars(v[r']_p\theta, t'_2 \cdot \gamma', \curvearrowright_{\underline{\vec{E}_\mathcal{B}}})$ analogously to before:
      A. $z \notin vars(v|_p)$. Then clearly $z\theta \equiv z$. In this case we have $z\theta$ not in $red\_vars(v[r']_p\theta, t'_2 \cdot \gamma', \curvearrowright_{\underline{\vec{E}_\mathcal{B}}})$ since otherwise, by the first property, we would have z in $red\_vars(v, t_2 \cdot \gamma, \curvearrowright_{\underline{\vec{E}_\mathcal{B}}})$, thus contradicting $red\_vars(v, t_2 \cdot \gamma, \curvearrowright_{\underline{\vec{E}_\mathcal{B}}}) \cap green\_vars(v, t_2) = \emptyset$.
      B. $z \in vars(v|_p)$. Then $z\theta$ not in $red\_vars(v[r']_p\theta, t'_2 \cdot \gamma', \curvearrowright_{\underline{\vec{E}_\mathcal{B}}})$ by the second property.
   ii. z' occurs in $r'\theta$.
      A. $z' \equiv z\theta$ for some variable $z \in vars(v|_p)$. As before, $z\theta$ not in $red\_vars(v[r']_p\theta, t'_2 \cdot \gamma', \curvearrowright_{\underline{\vec{E}_\mathcal{B}}})$ by the second property.
      B. $z' \not\equiv z\theta$ for any variable $z \in vars(v|_p)$. Then z' does not occur in $v[\,]_p\theta$ and hence z' not in $red\_vars(v[r']_p\theta, t'_2 \cdot \gamma', \curvearrowright_{\underline{\vec{E}_\mathcal{B}}})$.

   Therefore $red\_vars(v[r']_p\theta, t'_2 \cdot \gamma', \curvearrowright_{\underline{\vec{E}_\mathcal{B}}}) \cap green\_vars(v[r']_p\theta, t'_2) = \emptyset$.

2. $p \in P_{\mathcal{X}}(v, \sigma)$. In this case $v|_{p'}$ is a variable z for some prefix p' of p. Then $p = p' \cdot p''$ and $z\sigma|_{p''}$ is l. Now let σ' be the ground substitution with the same domain as σ but where $z\sigma' = z\sigma[r]_{p''}$ and $z'\sigma' = z'\sigma$ for all other variables z'. Let us see that $u \simeq v$ also fulfils the required conditions under this new substitution σ'. Here the key is to notice that $z \in red\_vars(v, t_2 \cdot \gamma, \curvearrowright_{\underline{\vec{E}_\mathcal{B}}})$. Then, given that $red\_vars(v, t_2 \cdot \gamma, \curvearrowright_{\underline{\vec{E}_\mathcal{B}}}) \cap vars(u) = \emptyset$, we have $u\sigma' \equiv u\sigma$ and hence condition 1 is preserved. Condition 2 is also preserved, since we have $v\sigma' \equiv t_2'\gamma'$, where $t_2' \equiv t_2[x_1]_{p_1} \ldots [x_n]_{p_n}$ and $\gamma' \equiv \gamma \cup \{x_1 \mapsto r, \ldots, x_n \mapsto r\}$ for some set of variables $x_1, \ldots, x_n$ and positions $p_1, \ldots, p_n$, and moreover $t_2 \cdot \gamma \overset{+}{\curvearrowright}_{\underline{\{l \to r\}}} t_2' \cdot \gamma'$, since $\{p_1, \ldots, p_n\} \subseteq red\_pos(t_2 \cdot \gamma, \overset{+}{\curvearrowright}_{\underline{\{l \to r\}}})$ thanks to $red\_vars(v, t_2 \cdot \gamma, \curvearrowright_{\underline{\vec{E}_\mathcal{B}}}) \cap green\_vars(v, t_2) = \emptyset$. Wrt. condition 3, we have $t_2'|_q \in \mathcal{X}$ or $t_2'|_q \equiv t_1|_q$ for all $q \in red\_pos(t_1 \cdot \beta, \curvearrowright_{\underline{\{l \to r\}}})$, since $t_2|_q \in \mathcal{X}$ or $t_2|_q \equiv t_1|_q$ for all $q \in red\_pos(t_1 \cdot \beta, \curvearrowright_{\underline{\{l \to r\}}})$, and $\{p_1, \ldots, p_n\} \subseteq red\_pos(t_1 \cdot \beta, \curvearrowright_{\underline{\{l \to r\}}})$ thanks to $red\_vars(v, t_2 \cdot \gamma, \curvearrowright_{\underline{\vec{E}_\mathcal{B}}}) \cap green\_vars(v, t_2) = \emptyset$. Finally, condition 4 is preserved since if $p'' \neq \lambda$ then $red\_vars(v, t_2' \cdot \gamma', \curvearrowright_{\underline{\vec{E}_\mathcal{B}}}) = red\_vars(v, t_2 \cdot \gamma, \curvearrowright_{\underline{\vec{E}_\mathcal{B}}})$ and $green\_vars(v, t_2') = green\_vars(v, t_2)$ and if, otherwise, $p'' = \lambda$ then $red\_vars(v, t_2' \cdot \gamma', \curvearrowright_{\underline{\vec{E}_\mathcal{B}}}) = red\_vars(v, t_2 \cdot \gamma, \curvearrowright_{\underline{\vec{E}_\mathcal{B}}}) \setminus \{z\}$ and $green\_vars(v, t_2') = green\_vars(v, t_2) \cup \{z\}$.

Once seen that the conditions are preserved, we conclude by induction hypothesis, since $\#\{q \in red\_pos(t_1 \cdot \beta, \curvearrowright_{\underline{\{l \to r\}}}) \mid t_2|_q \equiv t_1|_q\} > \#\{q \in red\_pos(t_1 \cdot \beta, \curvearrowright_{\underline{\{l \to r\}}}) \mid t_2'|_q \equiv t_1|_q\}$. □

Lemma 24. *Let $t_1 \cdot \rho$ be an abstracted term wrt. $\vec{E}_\mathcal{B}$. If there is an equation $u \simeq v$ in E such that, for some substitution σ, we have*

1. $u\sigma \curvearrowright_{\underline{\vec{E}_\mathcal{B}}} u\sigma[x]_{p_0} \cdot \varepsilon \overset{*}{\curvearrowright}_{\underline{\vec{E}_\mathcal{B}}} t_1 \cdot \rho$ *for some position p_0 not in $P_{\mathcal{B} \triangleleft u\sigma}(u\sigma)$,*
2. $v\sigma \equiv t_n\alpha$ *with* $t_1 \cdot \rho \overset{*}{\rightsquigarrow}_{\vec{E}_\mathcal{B}} t_n \cdot \alpha$, *and*
3. $red\_vars(v, t_n \cdot \alpha, \curvearrowright_{\underline{\vec{E}_\mathcal{B}}}) \cap (vars(u) \cup green\_vars(v, t_n)) = \emptyset$,

then there is an equation $u' \simeq v'$ in E such that, for some substitution σ', we have

1. $u'\sigma' \equiv u\sigma$,
2. $v'\sigma' \equiv t_m\beta$ *with* $t_n \cdot \alpha \overset{!}{\rightsquigarrow}_{\vec{E}_\mathcal{B}} t_m \cdot \beta$, *and*
3. $red\_vars(v', t_m \cdot \beta, \curvearrowright_{\underline{\vec{E}_\mathcal{B}}}) \cap (vars(u') \cup green\_vars(v', t_m)) = \emptyset$.

Proof. We will proceed by induction on $\overset{+}{\rightsquigarrow}_{\vec{E}_\mathcal{B}}$. If $t_n \cdot \alpha$ is irreducible wrt. $\rightsquigarrow_{\vec{E}_\mathcal{B}}$ then we are done: we choose u as u', v as v' and σ as σ'. If, otherwise, $t_n \cdot \alpha$ is reducible wrt. $\rightsquigarrow_{\vec{E}_\mathcal{B}}$, then there is a rule $l \to r$ in $\vec{E}_\mathcal{B}$ reducing $t_n \cdot \alpha$ wrt. $\curvearrowright_{\underline{\{l \to r\}}}$, with $l \not\equiv t_n\alpha$, and such that there is no rule $s \to t$ in $\vec{E}_\mathcal{B}$ with $l \rhd s$ reducing $t_n \cdot \alpha$ wrt. $\curvearrowright_{\underline{\{s \to t\}}}$. Therefore we can apply Lemma 23 with $t_1 \cdot \rho$ as $t_0 \cdot \alpha$, $t_n \cdot \alpha$ as $t_1 \cdot \beta$ and also as $t_2 \cdot \gamma$, and we obtain an equation $u' \simeq v'$ in E such that, under some substitution σ', fulfils the same conditions required to $u \simeq v$ under the substitution σ: note that we have $v'\sigma' \equiv t_3\delta$ with $t_n \cdot \alpha \rightsquigarrow_{\vec{E}_{\mathcal{B}}} t_3 \cdot \delta$, since $t_3|_q \in \mathcal{X}$ for all $q \in red\_pos(t_n \cdot \alpha, \curvearrowright_{\underline{\{l \to r\}}})$ implies $t_n \cdot \alpha \overset{\parallel}{\curvearrowright}_{\underline{\{l \to r\}}} t_3 \cdot \delta$. Then we conclude by induction hypothesis. $\square$

Lemma 25. *Let t be a term in $\mathcal{B}$. Let $\mathcal{S}$ be a non-empty subset of $\vec{E}_{\mathcal{B}}$ with non-overlapped left-hand sides and such that each rule in $\mathcal{S}$ reduces t at some position not in $P_{\mathcal{B}^{\lhd t}}(t)$. If there is an equation $u' \simeq v'$ in E such that, for some substitution σ', we have*

1. *$u'\sigma' \equiv t$,*
2. *$v'\sigma' \equiv t_m\beta$ with $t \overset{!}{\curvearrowright}_{\underline{\mathcal{S}}} t_1 \cdot \rho \overset{!}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} t_m \cdot \beta$, and*
3. *$red\_vars(v', t_m \cdot \beta, \curvearrowright_{\underline{\vec{E}_{\mathcal{B}}}}) \cap vars(u') = \emptyset$*

then $t_m\beta \succ t$ (i.e. $t_m\beta \to t \in \vec{E}_{\mathcal{B}}$).

Proof. We proceed by contradiction. Suppose that $t \succ t_m\beta$ (observe that, by Lemma 11, we have $t \not\equiv t_m\beta$). Since $t \in \mathcal{B}$, there is a rule $e = s \to t \in \mathcal{R}$. Wlog. we choose such an e with t as right-hand side to be minimal wrt. $\succ_E$. Then we have $t \notin \mathcal{B}^e$. Moreover s is irreducible by $\vec{E}_{\mathcal{B}} \setminus \{s \to t\}$ at topmost position by Lemma 8. We also have $s \not\rhd t$, since otherwise we would have $s \to t \succ_E l \to r$ for any rule $l \to r$ in $\mathcal{S}$, contradicting Lemma 17. Now let $s' \simeq t'$ be an equation such that (wlog.) $s'\sigma' \equiv s$ and $t'\sigma' \equiv t$. Note that, whatever terms are s' and t', if t' is a variable x then x cannot occur in s': on the one hand we clearly have $s' \not\equiv x$, since $s \succ t$; on the other hand we have $s' \not\rhd x$, since otherwise we would have $s \rhd t$. The following cases have to be considered:

1. t' is not a variable. Then there is an inference

$$\frac{u' \simeq v' \qquad s' \simeq t'}{(s' \simeq v')\theta}$$

   where $\theta = mgu(t', u')$, whose conclusion has an instance of the form $s \simeq t_m\beta$. Note that $s \simeq t_m\beta$ is not a tautology of the form $w \simeq w$, since $s \succ t \succ t_m\beta$. Then $e' = s \to t_m\beta \in \vec{E} \setminus \mathcal{R}$, given that s is irreducible by $\vec{E}_{\mathcal{B}} \setminus \{s \to t\}$ at topmost position and $\mathcal{R} \subseteq \vec{E}_{\mathcal{B}}$. Now we analyse the possible reasons why e' has not generated rule in $\mathcal{R}$. If e' failed condition 1 to generate rule then, by Lemma 4, also would have failed e, since $e = s \to t \succ_E s \to t_m\beta = e'$. Therefore e' must fail condition 2 to generate rule, i.e. there must be some rule $l \to r \in \mathcal{R}^{e'}$ reducing $t_m\beta$ at some position not in $P_{\mathcal{B}^{e'}}(t_m\beta)$. Then, by Lemma 4, $l \to r$ also reduces $t_m\beta$ at some position p not in $P_{\mathcal{B}}(t_m\beta)$. Two possibilities have to be considered:
   (a) $p \neq \lambda$. In this case we necessarily have $vars(t_m|_p) = \emptyset$, since $t_m \cdot \beta$ is irreducible wrt. $\rightsquigarrow_{\vec{E}_{\mathcal{B}}}$ and $l \to r \in \vec{E}_{\mathcal{B}}$. Now recall that $t \overset{!}{\curvearrowright}_{\underline{\mathcal{S}}} t_1 \cdot \rho \overset{!}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} t_m \cdot \beta$. Therefore, since $vars(t_m|_p) = \emptyset$, we have $t_m\beta|_p \equiv t|_p$. Moreover, by Lemma 15, there is a rewrite sequence $arg\_abs(t, \mathcal{R}) \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t'_m \cdot \beta'$ with $t'_m\beta' \equiv t_m\beta$. Now, since p not in $P_{\mathcal{B}}(t_m\beta)$, we have p not in $P_{\mathcal{B}}(t'_m\beta')$,

which implies p not in $P_{\mathcal{B}^{\triangleleft t}}(t)$. Indeed p is not in $P_{\mathcal{B}^e}(t)$ since, as stated before, t is not in $\mathcal{B}^e$. But this contradicts $e \in \mathcal{R}$ since, in this case, $l \to r$ is in R^e (since $e \succ e'$) and it reduces t at a position not in $P_{\mathcal{B}^e}(t)$.

(b) $p = \lambda$.

i. $v'\theta$ is not a variable. Then there is an inference

$$\frac{l' \simeq r' \qquad s'\theta \simeq v'\theta}{(s'\theta \simeq r')\theta'}$$

where $l' \simeq r'$ is an equation in E such that (wlog.) $l'\sigma \equiv l$ and $r'\sigma \equiv r$, where $\theta' = mgu(v'\theta, l')$, and whose conclusion has an instance of the form $s \simeq r$. Note that $s \simeq r$ is not a tautology of the form $w \simeq w$, since $s \succ t \succ t_m\beta \equiv l \succ r$. Then we have $s \to r \in \vec{E}_{\mathcal{B}}$, since $s \to r \in \vec{E}$ and $r \in \mathcal{B}$. But this contradicts $s \to t \in \vec{E}_{\mathcal{B}}$ by Lemma 8.

ii. $v'\theta$ is a variable. In this case v' must be a variable x too. Now, given that $\mathcal{S}$ is a non-empty subset of $\vec{E}_{\mathcal{B}}$ and $t \overset{!}{\curvearrowright}_{\overline{\mathcal{S}}} t_1 \cdot \rho \overset{!}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} t_m \cdot \beta$, we have $vars(t_m) \neq \emptyset$ and, hence, $x \in red\_vars(v', t_m \cdot \beta, \curvearrowright_{\overline{\vec{E}_{\mathcal{B}}}})$. Then, by condition 3, we have $x \notin vars(u')$ and, therefore, $v'\theta \equiv x$ and $x \notin vars(s'\theta)$. Then let σ'' be a substitution such that $x\sigma'' \equiv r$ and $y\sigma'' \equiv y\sigma'$ for all other variables y. We have $(s' \simeq v')\theta\sigma'' \equiv s \simeq r \in gnd(E)$, and we conclude as in the previous case.

2. t' is a variable x. In this case, as stated before, x does not occur in s'. Then let σ'' be a substitution such that $x\sigma'' \equiv t_m\beta$ and $y\sigma'' \equiv y\sigma'$ for all other variables y. We have $(s' \simeq t')\sigma'' \equiv s \simeq t_m\beta$. Then we proceed analogously to case 1 except for subcase 1b, where $s \simeq r$ is obtained again from $s' \simeq t'$ but under a different substitution.

□

Lemma 26. *Let t be a term in $\mathcal{B}$. Let $\mathcal{S}$ be a non-empty subset of $\vec{E}_{\mathcal{B}}$ with non-overlapped left-hand sides. If $t \overset{!}{\curvearrowright}_{\overline{\mathcal{S}}} t_1 \cdot \rho \overset{!}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} t_m \cdot \beta$ then $t_m\beta \equiv t$ or $t_m\beta \to t \in \vec{E}_{\mathcal{B}}$.*

Proof. Let D_1 denote the rewrite sequence $t \overset{!}{\curvearrowright}_{\overline{\mathcal{S}}} t_{n_1} \cdot \rho_{n_1} \overset{!}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} t_{m_1} \cdot \rho_{m_1}$ (recall that this rewrite sequence is unique up to permutation of non-overlapped steps by lemmas 12 and 13). Then, by Lemma 14, there is a rewrite sequence D_1' of the form $t \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t_{n_1'} \cdot \rho_{n_1'} \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t_{m_1'} \cdot \rho_{m_1'}$ which is equal to $D_1(\curvearrowright_{\overline{\vec{E}_{\mathcal{B}}}})$ up to permutation of non-overlapped steps, and such that $t_{n_1'}\rho_{n_1'} \equiv t$ and $t_{n_1'} \cdot \rho_{n_1'} \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t_{m_1'} \cdot \rho_{m_1'}$ consists of the steps in $D_1(\curvearrowright_{\overline{\vec{E}_{\mathcal{B}}}})$ which are applied at positions not in $P_{\mathcal{B}^{\triangleleft t}}(t)$. Then $t \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t_{n_1'} \cdot \rho_{n_1'}$ consists of the steps in $D_1(\curvearrowright_{\overline{\vec{E}_{\mathcal{B}}}})$ which are applied at positions in $P_{\mathcal{B}^{\triangleleft t}}(t)$.

Now, on the one hand, let $\mathcal{S}_n$ denote the subset of rules in $\mathcal{S}$ which reduce t at some position not in $P_{\mathcal{B}^{\triangleleft t}}(t)$ —which can possibly also reduce t at some position in $P_{\mathcal{B}^{\triangleleft t}}(t)$—. Observe that for all positions p in $red\_pos(t_{n_1'} \cdot \rho_{n_1'}, \curvearrowright_{\mathcal{S}_n})$ we have $vars(t_{n_1'}|_p) = \emptyset$, since rules in $\mathcal{S}_n$ have non-overlapped left-hand sides. Then $t_{n_1'} \cdot \rho_{n_1'} \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t_{m_1'} \cdot \rho_{m_1'}$ is equivalent to a rewrite sequence of the form $t_{n_1'} \cdot \rho_{n_1'} \overset{!}{\curvearrowright}_{\overline{\mathcal{S}_n}} t_{n_1''} \cdot \rho_{n_1''} \overset{!}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} t_{m_1''} \cdot \rho_{m_1''}$.

On the other hand, let $\mathcal{S}'_n$ denote the set of rules in $\vec{E}_\mathcal{B}$ which reduce $arg\_abs(t,\mathcal{R})$ wrt. $\rightsquigarrow_{\vec{E}_\mathcal{B}}$ and whose left-hand side is reducible by $\mathcal{S}$ but not by $\mathcal{S}_n$. And let $\mathcal{S}' = \mathcal{S}_n \cup \mathcal{S}'_n$. If $\mathcal{S}'$ is empty then we are done, since in this case t is irreducible by $\mathcal{S}$ at positions not in $P_{\mathcal{B}^{\triangleleft t}}(t)$ and $t_{n'_1} \cdot \rho_{n'_1}$ is irreducible wrt. $\rightsquigarrow_{\vec{E}_\mathcal{B}}$, and therefore $t_{n'_1} \cdot \rho_{n'_1} \equiv t_{m_1} \cdot \rho_{m_1}$. If, otherwise, $\mathcal{S}'$ is not empty, note that rules in $\mathcal{S}'$ have non-overlapped left-hand sides and each rule in $\mathcal{S}'$ reduces t at some position not in $P_{\mathcal{B}^{\triangleleft t}}(t)$. Then by lemmas 22, 24 and 25 we have there is a rewrite sequence D_2 of the form $t \overset{!}{\curvearrowright}_{\overline{\mathcal{S}'}} t_{n_2} \cdot \rho_{n_2} \overset{!}{\rightsquigarrow}_{\vec{E}_\mathcal{B}} t_{m_2} \cdot \rho_{m_2}$ and $t_{m_2}\rho_{m_2} \to t \in \vec{E}_\mathcal{B}$. Then, by Lemma 14, there is a rewrite sequence D'_2 of the form $t \overset{*}{\curvearrowright}_{\vec{E}_\mathcal{B}} t_{n'_2} \cdot \rho_{n'_2} \overset{*}{\curvearrowright}_{\vec{E}_\mathcal{B}} t_{m'_2} \cdot \rho_{m'_2}$ which is equal to $D_2(\curvearrowright_{\vec{E}_\mathcal{B}})$ up to permutation of non-overlapped steps, and such that $t_{n'_2} \cdot \rho_{n'_2} \equiv t$ and $t_{n'_2} \cdot \rho_{n'_2} \overset{*}{\curvearrowright}_{\vec{E}_\mathcal{B}} t_{m'_2} \cdot \rho_{m'_2}$ consists of the steps in $D_2(\curvearrowright_{\vec{E}_\mathcal{B}})$ which are applied at positions not in $P_{\mathcal{B}^{\triangleleft t}}(t)$. Observe that for all positions p in $red\_pos(t_{n'_2} \cdot \rho_{n'_2}, \curvearrowright_{\mathcal{S}'})$ we have $vars(t_{n'_2}|_p) = \emptyset$, since rules in $\mathcal{S}'$ have non-overlapped left-hand sides. Then $t_{n'_2} \cdot \rho_{n'_2} \overset{*}{\curvearrowright}_{\vec{E}_\mathcal{B}} t_{m'_2} \cdot \rho_{m'_2}$ is equivalent to a rewrite sequence of the form $t_{n'_2} \cdot \rho_{n'_2} \overset{!}{\curvearrowright}_{\overline{\mathcal{S}_n}} t_{n''_2} \cdot \rho_{n''_2} \overset{!}{\curvearrowright}_{\overline{\mathcal{S}'_n}} t_{n'''_2} \cdot \rho_{n'''_2} \overset{!}{\rightsquigarrow}_{\vec{E}_\mathcal{B}} t_{m''_2} \cdot \rho_{m''_2}$.

Finally observe that $t_{n''_1}\rho_{n''_1} \equiv t_{n''_2}\rho_{n''_2}$, since $t_{n'_1}\rho_{n'_1} \equiv t_{n'_2}\rho_{n'_2} \equiv t$. Moreover, by definition of $\mathcal{S}'_n$, we have that rewrite sequences $t_{n''_1} \cdot \rho_{n''_1} \overset{!}{\rightsquigarrow}_{\vec{E}_\mathcal{B}} t_{m''_1} \cdot \rho_{m''_1}$ and $t_{n''_2} \cdot \rho_{n''_2} \overset{!}{\curvearrowright}_{\overline{\mathcal{S}'_n}} t_{n'''_2} \cdot \rho_{n'''_2} \overset{!}{\rightsquigarrow}_{\vec{E}_\mathcal{B}} t_{m''_2} \cdot \rho_{m''_2}$ fulfil $t_{m''_1} \cdot \rho_{m''_1} \equiv t_{m''_2} \cdot \rho_{m''_2}$. Then we are done, since $t_{m''_1} \cdot \rho_{m''_1} \equiv t_{m'_1} \cdot \rho_{m'_1} \equiv t_{m_1} \cdot \rho_{m_1}$ and $t_{m''_2}\rho_{m''_2} \to t \in \vec{E}_\mathcal{B}$. □

Now, roughly we have to go from the return property with $\vec{E}_\mathcal{B}$ to the return property with $\mathcal{R}$. To this end we show that for some particular abstracted terms, the so-called *innermost reduced terms*, innermost rewrite sequences with $\curvearrowright_{\vec{E}_\mathcal{B}}$ are in fact rewrite sequences with $\curvearrowright_\mathcal{R}$ and preserve the innermost reduction property of the terms (Lemma 28). The proof of this lemma is based on Lemma 10.

Definition 17. *Let $s \cdot \gamma$ be an abstracted term wrt. $\mathcal{R}$. We say that $s \cdot \gamma$ is* innermost reduced *if there is no position $p \in red\_pos(s \cdot \gamma, \curvearrowright_{\vec{E}_\mathcal{B}})$ (i.e. redex in $s \cdot \gamma$) with $vars(s|_p) = \emptyset$ (i.e. which has never been reduced inside).*

Lemma 27. *Let $s \cdot \gamma$ be an abstracted term wrt. $\mathcal{R}$ such that is innermost reduced. If $s \cdot \gamma \curvearrowright_{\vec{E}_\mathcal{B}} t \cdot \delta$ then $t \cdot \delta$ is innermost reduced.*

Proof. Assume $t \cdot \delta$ is of the form $s[x]_p \cdot (\gamma \cup \{x \mapsto r\})$. Then, on the one hand, we have $vars(s|_{p'}) \neq \emptyset$ for any prefix p' of p. And, on the other hand, for any position q which is not overlapped with p we have $vars(s|_q) = vars(s[x]_p|_q)$, since $s|_q = s[x]_p|_q$. Therefore $s[x]_p \cdot (\gamma \cup \{x \mapsto r\})$ is innermost reduced, since $s \cdot \gamma$ is innermost reduced and, by definition of $\curvearrowright_{\vec{E}_\mathcal{B}}$, only positions above p or positions which are not overlapped with p can belong to $red\_pos(s[x]_p \cdot (\gamma \cup \{x \mapsto r\}), \curvearrowright_{\vec{E}_\mathcal{B}})$. □

Lemma 28. *Let $s \cdot \gamma$ be an abstracted term wrt. $\mathcal{R}$ which is innermost reduced and let D an innermost rewrite sequence of the form $s \cdot \gamma \overset{+}{\curvearrowright}_{\vec{E}_\mathcal{B}} t \cdot \delta$ by non-abstraction steps. Then D is a rewrite sequence with $\curvearrowright_\mathcal{R}$ and $t \cdot \delta$ is innermost reduced.*

Proof. By induction on the length of D. We have that D is of the form $s \cdot \gamma \curvearrowright_{\vec{E}_{\mathcal{B}}} s[x]_p \cdot (\gamma \cup \{x \mapsto r\}) \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t \cdot \delta$. Since it is innermost we have that there is no rule in $\vec{E}_{\mathcal{B}}$ reducing $s\gamma|_p$ at some position $p \cdot p'$ with $vars(s|_{p \cdot p'}) \neq \emptyset$ and $p' \neq \lambda$. But there is either no rule $\vec{E}_{\mathcal{B}}$ reducing $s\gamma|_p$ at some position $p \cdot p'$ with $vars(s|_{p \cdot p'}) = \emptyset$ and $p' \neq \lambda$, since $s \cdot \gamma$ is inner normalized. Therefore $s\gamma|_p$ can be reducible by $\vec{E}_{\mathcal{B}}$ at most at positions in $P_{\mathcal{X}}(s|_p)$, apart from λ. Now note that positions in $P_{\mathcal{X}}(s|_p)$ are $\mathcal{B}$-positions of $s\gamma|_p$, since $s \cdot \gamma$ is an abstracted term wrt. $\mathcal{R}$ and $\mathcal{B} = Right(\mathcal{R})$. So we have that $s\gamma|_p$ is irreducible by $\vec{E}_{\mathcal{B}}$ at any non-$\mathcal{B}$-position apart from λ, and therefore $s\gamma|_p \to r \in \mathcal{R}$ by Lemma 10. And by Lemma 27 we have that $s[x]_p \cdot (\gamma \cup \{x \mapsto r\})$ is innermost reduced. Now, if $s[x]_p \cdot (\gamma \cup \{x \mapsto r\}) \equiv t \cdot \delta$ then we are done. Otherwise we have $s[x]_p \cdot (\gamma \cup \{x \mapsto r\}) \overset{+}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t \cdot \delta$ and we conclude by induction hypothesis, since $s[x]_p \cdot (\gamma \cup \{x \mapsto r\})$ is an abstracted term wrt. $\mathcal{R}$ which is innermost reduced, and $s[x]_p \cdot (\gamma \cup \{x \mapsto r\}) \overset{+}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t \cdot \delta$ is a rewrite sequence by non-abstraction steps which is innermost and whose length is smaller than the length of D. ◻

Finally we obtain the return property of $\mathcal{B}$-terms for rewrite sequences using $\mathcal{R}$ by combining Lemmas 15 and 26: by carefully choosing the subset $\mathcal{S}$ of $\vec{E}_{\mathcal{B}}$ (and, in fact, of $\mathcal{R}$) for these lemmas, we obtain an abstracted term $t_1 \cdot \rho$ which is innermost reduced, and then we can show, by Lemma 28, that we have a return rewrite sequence using only $\mathcal{R}$. Finally, we can remove some steps of the resulting sequence showing that we have the sequence with $\mathcal{R}$ from $arg\_abs(t, \mathcal{R})$ (with the required initial step) to an abstraction of t again.

Theorem 1. *(return property) Let t be a term in $\mathcal{B}$. If $arg\_abs(t, \mathcal{R}) \curvearrowright_{\mathcal{R}} u \cdot \gamma$ then $u \cdot \gamma \overset{!}{\curvearrowright}_{\mathcal{R}} v \cdot \rho$ with $v\rho \equiv t$.*

Proof. Let $t' \cdot \delta$ denote $arg\_abs(t, \mathcal{R})$. If the step $arg\_abs(t, \mathcal{R}) \curvearrowright_{\mathcal{R}} u \cdot \gamma$ is an abstraction step (see definition 3) then $u \cdot \gamma$ is of the form $t'[x]_p \cdot (\delta \cup \{x \mapsto t'\delta|_p\})$ for some position p not in $P_{\mathcal{B}^{\triangleleft t}}(t)$ and $t'\delta|_p \in Right(\mathcal{R})$. Now observe that $t'\delta \equiv t$. Then $t|_p \in Right(\mathcal{R})$. But, since $Right(\mathcal{R}) = \mathcal{B}$ (see definition 8) and moreover $t \in \mathcal{B}$ and p not in $P_{\mathcal{B}^{\triangleleft t}}(t)$, we must have $p = \lambda$. Therefore in this case $u \cdot \gamma$ is of the form $x \cdot \{x \mapsto t\}$, and we are done.

If, otherwise, the step $arg\_abs(t, \mathcal{R}) \curvearrowright_{\mathcal{R}} u \cdot \gamma$ is not an abstraction step then $u \cdot \gamma$ is of the form $t'[x]_p \cdot (\delta \cup \{x \mapsto r\})$ for some position p not in $P_{\mathcal{B}^{\triangleleft t}}(t)$, $t'\delta|_p \equiv l$ and $l \to r \in \mathcal{R}$. In this case let $\mathcal{S}_n$ denote the set of rules $l' \to r'$ in $\vec{E}_{\mathcal{B}}$ such that $l' \to r'$ reduces t. And let $\mathcal{S}$ denote the set of rules $l' \to r'$ in $\mathcal{S}_n$ whose left-hand side is minimal wrt. $\rhd$ in $Left(\mathcal{S}_n)$. Then rules in $\mathcal{S}$ have non-overlapped left-hand sides since for any rule $l' \to r'$ in $\vec{E}_{\mathcal{B}}$ we have that l' is irreducible by $\vec{E}_{\mathcal{B}} \setminus \{l' \to r'\}$ at topmost position by Lemma 8. Moreover $\mathcal{S} \subseteq \mathcal{R}$ by Lemma 10. And moreover $\mathcal{S}$ is not empty, since the step $arg\_abs(t, \mathcal{R}) \curvearrowright_{\mathcal{R}} u \cdot \gamma$ is not an abstraction step. Now let D denote the rewrite sequence $t \overset{!}{\curvearrowright}_{\mathcal{S}} t_1 \cdot \rho \overset{!}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} t_m \cdot \beta$ (recall that this rewrite sequence is unique up to permutation of non-overlapped steps by lemmas 12 and 13). And let D_1 denote the subsequence $t \overset{!}{\curvearrowright}_{\mathcal{S}} t_1 \cdot \rho$ and D_2 the subsequence $t_1 \cdot \rho \overset{!}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} t_m \cdot \beta$. We have that $t_m\beta \equiv t$ or $t_m\beta \to t \in \vec{E}_{\mathcal{B}}$ by

Lemma 26. Now, on the one hand, by Lemma 15 we have that there is a rewrite sequence D' of the form $arg\_abs(t,\mathcal{R}) \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t'_m \cdot \beta'$ with $t'_m\beta' \equiv t_m\beta$ consisting of the steps in $D(\curvearrowright_{\vec{E}_{\mathcal{B}}})$ which are applied at positions not in $P_{\mathcal{B}^{\triangleleft t}}(t)$. On the other hand we have that $t_1 \cdot \rho$ is an abstracted term wrt. $\mathcal{R}$, since $\mathcal{S} \subseteq \mathcal{R}$. And it is innermost reduced (see definition 17) by definition of $\mathcal{S}$. Then, by Lemma 28, we have that $D_2(\curvearrowright_{\vec{E}_{\mathcal{B}}})$ is a rewrite sequence with $\curvearrowright_{\mathcal{R}}$ and its last term is innermost reduced. Now recall that D_1 is a rewrite sequence with $\curvearrowright_{\mathcal{S}}$ and $\mathcal{S} \subseteq \mathcal{R}$. Therefore $D(\curvearrowright_{\vec{E}_{\mathcal{B}}})$ is a rewrite sequence with $\curvearrowright_{\mathcal{R}}$, and so is D'. Now let $t''_m \cdot \beta''$ denote the last term in D_2. We clearly have $t''_m \cdot \beta'' \equiv t'_m \cdot \beta' \equiv t_m \cdot \beta$. Therefore since, as we have seen, it is innermost reduced, if $t_m\beta \to t \in \vec{E}_{\mathcal{B}}$ then the rewrite sequence $t''_m \cdot \beta'' \curvearrowright_{\{t_m\beta \to t\}} x \cdot \{x \mapsto t\}$ is a rewrite sequence with $\curvearrowright_{\mathcal{R}}$ by Lemma 28, and hence $t_m\beta \to t \in \mathcal{R}$. Finally let D'' denote the rewrite sequence D' itself if $t_m\beta \equiv t$, or D' followed by a last step with $\curvearrowright_{\{t_m\beta \to t\}}$ otherwise. Note that in either case the last term in D'' is irreducible wrt. $\curvearrowright_{\mathcal{R}}$. This is clearly the case for $x \cdot \{x \mapsto t\}$ and, for the case that $t_m\beta \equiv t$, recall that $t_m \cdot \beta$ is irreducible wrt. $\rightsquigarrow_{\vec{E}_B}$, and hence it could only be reduced wrt. $\curvearrowright_{\vec{E}_{\mathcal{B}}}$ (and hence wrt. $\curvearrowright_{\mathcal{R}}$) at topmost position, but this is not possible by Lemma 7. To conclude we show that a non-abstraction step with $l \to r$ at position p is present in D'', and that D'' can be rearranged to make this be the first step. Observe that $D(\curvearrowright_{\vec{E}_{\mathcal{B}}})$ does not have abstraction steps by definition of $\curvearrowright_{\overline{\mathcal{R}}}$ and $\curvearrowright_{\underline{\mathcal{R}}}$, since D_1 is a rewrite sequence with $\curvearrowright_{\overline{\mathcal{R}}}$ and $D_2(\curvearrowright_{\vec{E}_{\mathcal{B}}})$ is a rewrite sequence with $\curvearrowright_{\underline{\mathcal{R}}}$. Therefore neither D' nor D'' have abstraction steps. Now recall that D' consists of the steps in $D(\curvearrowright_{\vec{E}_{\mathcal{B}}})$ which are applied at positions not in $P_{\mathcal{B}^{\triangleleft t}}(t)$. If $l \to r \in \mathcal{S}$ then a non-abstraction step with $l \to r$ at position p is clearly present in D' and hence in D''. If $l \to r \notin \mathcal{S}$ then l must be reducible by some rule in $\mathcal{S}$. But rules in $\mathcal{S}$ can only reduce l at $\mathcal{B}$-positions by Lemma 10, since $l \to r \in \mathcal{R}$. Now observe that we necessarily have $t \rhd l$ by Lemma 6. So for any $\mathcal{B}$-position p' of l which is reducible by $\mathcal{S}$ we have that $p \cdot p'$ is in $P_{\mathcal{B}^{\triangleleft t}}(t)$, since $t|_p \equiv l$. Then by Lemma 14 we have that there is a rewrite sequence D''' of the form $t \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t_n \cdot \alpha \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t'''_m \cdot \beta'''$ which is equal to $D(\curvearrowright_{\vec{E}_{\mathcal{B}}})$ up to permutation of non-overlapped steps, and such that $t_n\alpha \equiv t$ and $t_n \cdot \alpha \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t'''_m \cdot \beta'''$ consists of the steps in $D(\curvearrowright_{\vec{E}_{\mathcal{B}}})$ which are applied at positions not in $P_{\mathcal{B}^{\triangleleft t}}(t)$. Therefore we have that p is a valid position of t_n and $vars(t_n|_p) \neq \emptyset$, since p is a position not in $P_{\mathcal{B}^{\triangleleft t}}(t)$ and $t|_p$ is reducible at $\mathcal{B}$-positions by $\mathcal{S}$. So $l \to r$ reduces $t_n \cdot \alpha$ at position p wrt. $\curvearrowright_{\vec{E}_{\mathcal{B}}}$. Therefore in any innermost rewrite sequence with $\curvearrowright_{\vec{E}_{\mathcal{B}}}$ from $t_n \cdot \alpha$ there must be a step with $l \to r$ at position p, since l is irreducible by $\vec{E}_{\mathcal{B}}$ at any non-$\mathcal{B}$ position apart from λ by Lemma 10, and there are no two rules with the same left-hand side in $\vec{E}_{\mathcal{B}}$ by Lemma 8. So a step with $l \to r$ occurs in $D_2(\curvearrowright_{\vec{E}_{\mathcal{B}}})$ at position p, and hence also in D' and D''. Moreover D'' can be rearranged to make this be the first step, since $t|_p \equiv l$ and D'' is innermost. This gives us a rewrite sequence of the form $arg\_abs(t,\mathcal{R}) \curvearrowright_{\mathcal{R}} u \cdot \gamma \overset{!}{\curvearrowright}_{\mathcal{R}} v \cdot \rho$ with $v\rho \equiv t$, and hence we are done. $\square$

4.2 Rewrite Proofs for the equational case

First we prove the confluence of the rewrite relation $\overset{*}{\curvearrowright}_{\mathcal{R}}$, which is essential for showing that we have $\mathcal{B}$-rewrite proofs for all consequences of the set of equations.

Using the return property we can show that if we reduce an abstraction of a left-hand side in $\mathcal{R}$ (not at top position) then there is a $\mathcal{B}$-rewrite sequence going back to (another abstraction of) the left-hand side again. The proof is base on the fact that left-hand sides can only be reducible by the other rules at $\mathcal{B}$-positions.

Lemma 29. *Let $s \cdot \gamma$ be an abstracted term such that $s\gamma \in Left(\mathcal{R})$. If there is a position $p \neq \lambda$, such that $s|_p\gamma \equiv l$ for some rule $l \to r$ in $\mathcal{R}$, then $s[x]_p \cdot (\gamma \cup \{x \mapsto r\} \overset{+}{\curvearrowleft}_{\mathcal{R}} s' \cdot \gamma'$, where s' is not a variable and $s\gamma \equiv s'\gamma'$.*

Proof. Since $s\gamma$ is in $Left(\mathcal{R})$, by lemma 10, $s\gamma$ is irreducible at non-$\mathcal{B}$-positions different from λ. Therefore, since $p \neq \lambda$, there is a position $q \neq \lambda$ with $q \cdot p' = p$ for some $p' \neq \lambda$, s.t. $s|_q\gamma$ is a $\mathcal{B}$-term. We take the deepest such position. Then, $s|_q \cdot \gamma$ is $arg\_abs(s|_q\gamma, \mathcal{R})$ and $s|_q \cdot \gamma \curvearrowright_{\mathcal{R}} s|_q[x]_{p'} \cdot (\gamma \cup \{x \mapsto r\})$. Therefore, by theorem 1, we have that $s|_q[x]_{p'} \cdot (\gamma \cup \{x \mapsto r\}) \overset{!}{\curvearrowleft}_{\mathcal{R}} v \cdot \rho$ with $v\rho \equiv s|_q\gamma$, which implies $s[x]_p \cdot (\gamma \cup \{x \mapsto r\}) \overset{+}{\curvearrowleft}_{\mathcal{R}} s[v]_q \cdot (\gamma \cup \rho)$ and $s[v]_q(\gamma \cup \rho) \equiv s\gamma$. Hence, with $s' = s[v]_q$ and $\gamma' = (\gamma \cup \rho)$, it holds. □

¿From the previous lemma we know that we can close the only potential "picks" that we may have with $\curvearrowright_{\mathcal{R}}$. Therefore, local confluence follows.

Lemma 30. *(Local confluence) Let $s \cdot \gamma$ be an abstracted term. If $t \cdot \gamma_1 \curvearrowleft_{\mathcal{R}} s \cdot \gamma \curvearrowright_{\mathcal{R}} u \cdot \gamma_2$ then $t \cdot \gamma_1 \overset{*}{\curvearrowright}_{\mathcal{R}} v \cdot \delta \overset{*}{\curvearrowleft}_{\mathcal{R}} u \cdot \gamma_2$, for some abstracted term $v \cdot \delta$.*

Proof. Let $l_1 \to r_1$ be the rule that is applied at some position p_1 in the step $s \cdot \gamma \curvearrowright_{\mathcal{R}} t \cdot \gamma_1$ and let $l_2 \to r_2$ be the rule that is applied at some position p_2 in the step $s \cdot \gamma \curvearrowright_{\mathcal{R}} u \cdot \gamma_2$. Then $t = s[x]_{p_1}$ for some fresh variable x and $\gamma_1 = \gamma \cup \{x \mapsto r_1\}$ and $u = s[y]_{p_2}$ for some fresh variable y and $\gamma_2 = \gamma \cup \{y \mapsto r_2\}$

There are two cases to be considered.

1. There is no overlapping between p_1 and p_2. We can close easily the diamond, applying $l_2 \to r_2$ at position p_2 in $t \cdot \gamma_1$ and $l_1 \to r_1$ at position p_1 in $u \cdot \gamma_2$ obtaining $v = s[x]_{p_1}[y]_{p_2}$ and $\delta = \gamma \cup \{x \mapsto r_1\} \cup \{y \mapsto r_2\}$.
2. There is an overlapping between p_1 and p_2. Assume that p_2 is below p_1, and hence $p_2 = p_1 \cdot p_3$ for some p_3. Then, since the relation $\curvearrowright_{\mathcal{R}}$ is monotonic, it is enough to show that we can close the diamond starting from $s|_{p_1} \cdot \gamma$, i.e. we have to show that there is some abstracted term $v' \cdot \delta'$ s.t. $x \cdot \{x \mapsto r_1\} \overset{*}{\curvearrowright}_{\mathcal{R}} v' \cdot \delta' \overset{*}{\curvearrowleft}_{\mathcal{R}} s|_{p_1}[y]_{p_2} \cdot (\gamma \cup \{y \mapsto r_2\})$.
   By lemma 29, since $s|_{p_1}\gamma \in Left(\mathcal{R})$ we have that $s|_{p_1}[y]_{p_2} \cdot (\gamma \cup \{y \mapsto r_2\} \overset{+}{\curvearrowleft}_{\mathcal{R}} s' \cdot \gamma'$, where s' is not a variable and $s|_{p_1}\gamma \equiv s'\gamma'$. Therefore, applying finally $l_1 \to r_1$ on $s' \cdot \gamma'$ we have that $x \cdot \{x \mapsto r_1\} \overset{*}{\curvearrowleft}_{\mathcal{R}} s|_{p_1}[y]_{p_2} \cdot (\gamma \cup \{y \mapsto r_2\})$, which implies that taking $v' = x$ and $\delta' = \{x \mapsto r_1\}$ it holds. □

Since $\curvearrowright_{\mathcal{R}}$ is terminating and locally confluent, it is confluent.

Theorem 2. *(Confluence) $\curvearrowright_{\mathcal{R}}$ is confluent.*

Now we prove that all equations in E have a $\mathcal{B}$-rewrite proof using $\mathcal{R}$, which also implies that $\mathcal{R}^*$ is a model for E.

First, we show that there is a $\mathcal{B}$-rewrite proof for all instances which are ireducible (at non-$\mathcal{B}$-positions) in the substitution part. A substitution σ is irreducible by $\mathcal{R}$ at non-$\mathcal{B}$-positions in a term t if, for all variables $x \in Dom(\sigma)$, $x\sigma$ is irreducible by $\mathcal{R}$ at non-$\mathcal{B}$-positions or x occurs in t only at $\mathcal{B}$-positions of $t\sigma$. A substitution σ is irreducible by $\mathcal{R}$ at non-$\mathcal{B}$-positions in an equation $s \simeq t$ if σ is irreducible by $\mathcal{R}$ at non-$\mathcal{B}$-positions in s and t. This irreducibility property is roughly inherited by the conclusion in paramodulation inferences with rules in $\mathcal{R}$.

Lemma 31. *Let $t \cdot \gamma$ be an abstracted term wrt $\mathcal{R}$. If $t \cdot \gamma$ is irreducible by $\curvearrowright_{\mathcal{R}}$ then $t\gamma$ is irreducible by $\mathcal{R}$ at non-$\mathcal{B}$-positions.*

Proof. It holds trivially since the range of γ only includes right-hand side terms of $\mathcal{R}$, i.e. $\mathcal{B}$-terms, and hence since $t \cdot \gamma$ is irreducible, $t\gamma$ can only be reducible in γ, i.e. in a $\mathcal{B}$-position. □

Lemma 32. *Let $s \simeq t$ be an equation in E and σ a substitution irreducible at non-$\mathcal{B}$-positions in $s \simeq t$. If $u_1 \cdot \gamma_1$ is an abstraction of $s\sigma$ wrt. $\mathcal{R}$ and $u_2 \cdot \gamma_2$ is an abstraction of $t\sigma$ wrt. $\mathcal{R}$ then $u_1 \cdot \gamma_1 \overset{*}{\curvearrowright}_{\mathcal{R}} u \cdot \gamma \overset{*}{\curvearrowleft}_{\mathcal{R}} u_2 \cdot \gamma_2$ for some abstracted term $u \cdot \gamma$.*

Proof. We proceed by induction on $|u_1|_v + |u_2|_v$.

1. If either $u_1 \cdot \gamma_1$ or $u_2 \cdot \gamma_2$ has a $\mathcal{B}$-term in a non-variable position in u_1 or u_2 (respectively) then we abstract in this position. We show it for u_1 (for u_2 it is analogous). Let p be a non-variable position in u_1 s.t. $u_1|_p\gamma_1$ is in $Right(\mathcal{R})$. Then we have that $u_1 \cdot \gamma_1 \curvearrowright_{\mathcal{R}} u_1[x]_p \cdot \gamma_1'$, where $\gamma_1' = (\gamma_1 \cup \{x \mapsto u_1|_p\gamma_1\})$, and, by induction hypothesis, it holds that $u_1[x]_p \cdot \gamma_1' \overset{*}{\curvearrowright}_{\mathcal{R}} u \cdot \gamma \overset{*}{\curvearrowleft}_{\mathcal{R}} u_2 \cdot \gamma_2$ for some abstracted term $u \cdot \gamma$.
2. If neither $u_1 \cdot \gamma_1$ nor $u_2 \cdot \gamma_2$ has a $\mathcal{B}$-term in a non-variable position in u_1 or u_2 (respectively) then there are two possibilities. If $u_1 \cdot \gamma_1 \equiv u_2 \cdot \gamma_2$ then we are done. Otherwise, since $u_1 \cdot \gamma_1$ is a top abstraction of $s\sigma$ and $u_2 \cdot \gamma_2$ is a top abstraction of $t\sigma$, and they are not equivalent, we have that $s\sigma \not\equiv t\sigma$. Assume $s\sigma \succ t\sigma$.
   If $s\sigma \to t\sigma$ is in $\mathcal{R}$, then $t\sigma \in \mathcal{B}$, and hence u_2 is a variable x and γ_2 is $\{x \mapsto t\sigma\}$ and, $u_1 \cdot \gamma_1 \curvearrowright_{\mathcal{R}} x \cdot \{x \mapsto t\sigma\}$. Therefore, taking $u = x$ and $\gamma = \gamma_2$ we are done.
   Otherwise, if $s\sigma \to t\sigma$ is not in $\mathcal{R}$, then, by lemma 4 and the definition of $\mathcal{R}$, either $s\sigma$ or $t\sigma$ should be reducible by $\mathcal{R}$ at positions not in $P_{\mathcal{B}}(t)$, and hence in a non-variable position in u_1 or u_2 (respectively). We consider the case in which $s\sigma$ is reducible (the case for $t\sigma$ is analogous). Let p be a non-variable position in u_1 s.t. there is a rule $l \to r$ in $\mathcal{R}$ with $u_1|_p\gamma_1 \equiv l$. Then,

we have that $u_1 \cdot \gamma_1 \curvearrowright_{\mathcal{R}} u_1' \cdot \gamma_1'$ where $u_1' = u_1[x]_p$ and $\gamma_1' = \gamma_1 \cup \{x \mapsto r\}$. On the other hand, since σ is irreducible at non-$\mathcal{B}$-positions in $s \simeq t$, p is a non-variable position of s and if $l' \simeq r'$ is an equation of E such that $l'\sigma' \equiv l$ and $r'\sigma' \equiv r$ for some substitution σ', then we have and inference by ordered paramodulation

$$\frac{l' \simeq r' \qquad s \simeq t}{(s[r']_p \simeq t)\theta}$$

where $\theta = mgu(s|_p, l')$ and $\sigma'' = \sigma \cup \sigma'$ is a substitution such that σ'' is irreducible at non-$\mathcal{B}$-positions in $(s[r']_p \simeq t)\theta$ (note that $r'\theta\sigma'' \equiv r \in \mathcal{B}$) and $l'\theta\sigma'' \equiv l \succ r \equiv r'\theta\sigma''$. Moreover, since $s[r']_p\theta\sigma'' \equiv s\sigma[r]_p$ and $t\theta\sigma'' \equiv t\sigma$, we have that $u_1' \cdot \gamma_1'$ is an abstraction of $s[r']_p\theta\sigma''$ and $u_2 \cdot \gamma_2$ is an abstraction of $t\theta\sigma''$. Finally, since $|u_1|_v > |u_1'|_v$, by induction hypothesis, $u_1 \cdot \gamma_1 \curvearrowright_{\mathcal{R}} u_1' \cdot \gamma_1' \overset{*}{\curvearrowright}_{\mathcal{R}} u \cdot \gamma \overset{*}{\curvearrowleft}_{\mathcal{R}} u_2 \cdot \gamma_2$ for some abstracted term $u \cdot \gamma$. □

Second, by first normalizing the substitution wrt. $\overset{*}{\curvearrowleft}_{\mathcal{R}}$, which gives us an irreducible instance, we have that there are $\mathcal{B}$-rewrite proofs for all instances.

Lemma 33. *Let $s \simeq t$ be an equation in $gnd(E)$. Then $s \overset{*}{\curvearrowright}_{\mathcal{R}} u \cdot \gamma \overset{*}{\curvearrowleft}_{\mathcal{R}} t$ for some abstracted term $u \cdot \gamma$.*

Proof. Let $s \simeq t$ be an instance of an equation $s' \simeq t'$ in E, with $s \equiv s'\sigma$ and $t \equiv t'\sigma$ for some substitution σ. First of all we normalize σ, i.e. we define a new substitution σ' s.t. $x\sigma' \equiv w\gamma_w$ if $x\sigma \overset{!}{\curvearrowright}_{R} w \cdot \gamma_w$. Then there is some equation in $gnd(E)$ of the form $s'\sigma' \simeq t'\sigma'$, s.t. $s \overset{*}{\curvearrowright}_{\mathcal{R}} u_1 \cdot \gamma_1$ and $t \overset{*}{\curvearrowright}_{\mathcal{R}} u_2 \cdot \gamma_2$ for some $u_1 \cdot \gamma_1$ and $u_2 \cdot \gamma_2$ abstracted terms wrt. $\mathcal{R}$ of $s'\sigma'$ and $t'\sigma'$ respectively (these abstracted terms are easily obtained from the normalization of the substitution, i.e. abstracting each γ_w).

Then, by lemma 32, we have $u_1 \cdot \gamma_1 \overset{*}{\curvearrowright}_{\mathcal{R}} u \cdot \gamma \overset{*}{\curvearrowleft}_{\mathcal{R}} u_2 \cdot \gamma_2$, which allows us to conclude composing the rewriting steps. □

The following theorem follows directly from the previous lemma. Note that rewriting with $\curvearrowright_{\mathcal{R}}$ provides also a rewrite sequence with $\rightarrow_{\mathcal{R}}$.

Theorem 3. *$\mathcal{R}^* \models E$.*

Now, together with confluence of $\overset{*}{\curvearrowright}_{\mathcal{R}}$, we can show that all consequences of E have a $\mathcal{B}$-rewrite proof.

Lemma 34. *$E \models s \simeq t$ implies $s \overset{*}{\curvearrowright}_{\mathcal{R}} u \cdot \gamma \overset{*}{\curvearrowleft}_{\mathcal{R}} t$ for some abstracted term $u \cdot \gamma$.*

Proof. Since $E \models s \simeq t$, we can go from s to t using the equations of $gnd(E)$. Then there is a sequence of terms $s_0, s_1 \ldots, s_{n+1}$ such that $s_0 = s$, $s_{n+1} = t$ and for every $i \in \{1 \ldots n-1\}$ we have $s_i = C[e_1]$ and $s_{i+1} = C[e_2]$ for some context C and $e_1 \simeq e_2 \in gnd(E)$. By lemma 33 and monotonicity of $\overset{*}{\curvearrowright}_{\mathcal{R}}$, for every $i \in \{1 \ldots n-1\}$ we have $s_i \overset{*}{\curvearrowright}_{\mathcal{R}} u_i \cdot \gamma_i \overset{*}{\curvearrowleft}_{\mathcal{R}} s_{i+1}$ for some abstracted term $u_i \cdot \gamma_i$. Finally, by confluence of $\overset{*}{\curvearrowright}_{\mathcal{R}}$, we have that $s = s_0 \overset{*}{\curvearrowright}_{\mathcal{R}} u \cdot \gamma \overset{*}{\curvearrowleft}_{\mathcal{R}} s_{n+1} = t$ for some abstracted term $u \cdot \gamma$. □

This result implies that we have produced a Knuth-Bendix completion procedure for $\mathcal{B}$-rewriting, i.e. for ordered rewriting with well-founded orderings.

5 The Horn case

In this section we generalise the results of the previous section to Horn clauses. In the following inference system it is assumed that in each clause with a non-empty antecedent one of the negative equations, the one that is written underlined, has been *selected* (see, e.g., [BG98]). In the Horn case this leads to positive unit strategies: left premises of paramodulations are unit clauses, and the only inferences involving non-unit clauses are equality resolution or paramodulation left on its selected equation.

Definition 18. *The inference system $\mathcal{H}$ for Horn clauses with respect to the well-founded ordering $\succ$ is defined as follows:*

paramodulation right:

$$\frac{\rightarrow l \simeq r \qquad \rightarrow s \simeq t}{\rightarrow (s[r]_p \simeq t)\sigma} \qquad \text{where } \sigma = mgu(l, s|_p)$$

paramodulation left:

$$\frac{\rightarrow l \simeq r \qquad \Gamma, \underline{s \simeq t} \rightarrow \Delta}{(\Gamma, s[r]_p \simeq t \rightarrow \Delta)\sigma} \qquad \text{where } \sigma = mgu(l, s|_p)$$

equality resolution:

$$\frac{\Gamma, \underline{s \simeq t} \rightarrow \Delta}{(\Gamma \rightarrow \Delta)\sigma} \qquad \text{where } \sigma = mgu(s, t)$$

where moreover in both paramodulation rules l is maximal in its premise (that is, for some ground substitution θ, it holds that $l\sigma\theta \succ r\sigma\theta$), and $s|_p$ is not a variable except in the paramodulation right inference when $x \in vars(t)$, $p = \lambda$ and l is also a variable (as it happens in the equational case).

Definition 19. *Let S be a set of Horn clauses. We denote by E_S the subset of all positive unit clauses in S, that is, $E_S = \{\ s \simeq t \ \mid \rightarrow s \simeq t \in S\ \}$, and we denote by $\mathcal{R}_S$ the set of rules generated by E_S as in Definition 8 (i.e. $\mathcal{R}_S = \mathcal{R}_{E_S}$).*

First, it is shown that if S does not contain the empty clause then $\mathcal{R}_S^*$ a model for all irreducible (at non-$\mathcal{B}$-positions) instances of clauses in S.

A substitution σ is irreducible by $\mathcal{R}_S$ at non-$\mathcal{B}$-positions in a clause $\Gamma \rightarrow \Delta$ if σ is irreducible at non-$\mathcal{B}$-positions in all equation $s \simeq t$ ocurring in Γ or Δ.

Lemma 35. *Let S be a set of Horn clauses closed under $\mathcal{H}$. Then $\mathcal{R}_S \models C\sigma$ for all clauses C in S and substitutions σ that are irreducible by $\mathcal{R}_S$ at non-$\mathcal{B}$-positions in C.*

Proof. We prooceed by minimal counterexample. Let $C' \cdot \gamma$ be the abstracted clause wrt. $\mathcal{R}_S$ with minimal $|C'|_v$ s.t. $C' \cdot \gamma$ is an abstraction of a clause $C\sigma$ with C in S and σ irreducible by $\mathcal{R}_S$ at non-$\mathcal{B}$-positions in C.

Since E_S is closed under ordered paramodulation, by Theorem 3, we have $R_S^* \models E_S$, and therefore $C\sigma$ cannot be a positive unit clause. Then C is of the form $\Gamma, \underline{s \simeq t} \rightarrow \Delta$ and C' is of the form $\Gamma', \underline{s' \simeq t'} \rightarrow \Delta'$. Two cases have to be considered:

1. If $s\sigma \equiv t\sigma$ then there is an inference by equality resolution
$$\frac{\Gamma, \underline{s \simeq t} \rightarrow \Delta}{(\Gamma \rightarrow \Delta)\theta}$$
whose conclusion D has an instance $D\sigma$ such that $(\Gamma' \rightarrow \Delta') \cdot \gamma$ is an abstraction of $D\sigma$ with $|C'|_v > |(\Gamma' \rightarrow \Delta')|_v$ and moreover, σ is irreducible by $\mathcal{R}_S$ at non-$\mathcal{B}$-positions in D and $R_S^* \not\models D\sigma$, which is a contradiction.
2. If $s\sigma \not\equiv t\sigma$ then, since $R_S^* \not\models C\sigma$, we have $R_S^* \models s\sigma \simeq t\sigma$, and, by Theorem 3 and Lemma 34, there must be a $\mathcal{B}$-rewrite proof of $s\sigma \simeq t\sigma$ by $\mathcal{R}_S$. Moreover, since $s\sigma \overset{*}{\curvearrowright}_{\mathcal{R}_S} s' \cdot \gamma$ and $t\sigma \overset{*}{\curvearrowright}_{\mathcal{R}_S} t' \cdot \gamma$ (using abstaction steps), by confleunece of $\curvearrowright_{\mathcal{R}_S}$, there is some abstracted term $u \cdot \delta$ such that $s' \cdot \gamma \overset{*}{\curvearrowright}_{\mathcal{R}} u \cdot \delta \overset{*}{\curvearrowleft}_{\mathcal{R}} t' \cdot \gamma$. Since $s'\gamma \not\equiv t'\gamma$, either $s' \cdot \gamma$ or $t' \cdot \gamma$ is reducible by $\mathcal{R}_S$ (note that the $\mathcal{B}$-rewrite proof cannot include only abstraction steps). Assume that $s' \cdot \gamma$ is reducible (the other case is analogous) at some non-variable position p in s'. By irreducibility of the substitution σ at non-$\mathcal{B}$-positions of C, p should be a non-variable position in s, and hence there is an inference by paramodulation left:
$$\frac{\rightarrow l \simeq r \qquad \Gamma, \underline{s \simeq t} \rightarrow \Delta}{(\Gamma, s[r]_p \simeq t \rightarrow \Delta)\theta}$$
whose conclusion D has an instance $D\sigma$ such that $(\Gamma', s'[x]_p \simeq t' \rightarrow \Delta') \cdot (\gamma \cup \{x \mapsto r\sigma\}$ is an abstraction of $D\sigma$ with $|C'|_v > |(\Gamma', s'[x]_p \simeq t' \rightarrow \Delta')|_v$ and moreover, σ is irreducible by $\mathcal{R}_S$ at non-$\mathcal{B}$-positions in D (note that all variables in $r\sigma$ is a $\mathcal{B}$-term) and $R_S^* \not\models D\sigma$, which is a contradiction. $\square$

Now, as in the equational case, by first normalizing the substitution, we can prove the main theorem.

Theorem 4. *(refutation completeness of $\mathcal{H}$ for Horn clauses)*
Let S be a set of Horn clauses closed under $\mathcal{H}$. Then $\square \in S$ if, and only if, S is unsatisfiable.

Proof. The left to right implication is trivial. For the other one, let S be a set of clauses closed under $\mathcal{H}$ such that $\square \notin S$, we prove that $\mathcal{R}_S^*$ is a model of S, i.e., $\mathcal{R}_S^* \models C\sigma$ for all clause C in S and substitution σ. Let σ' be a substitution such that $x\sigma' \equiv w\gamma_w$ if $x\sigma \overset{!}{\curvearrowright}_{R_S} w \cdot \gamma_w$. Then $\mathcal{R}_S \cup \{C\sigma'\} \models C\sigma$ and, since σ' is irreducible by $\mathcal{R}_S$ at non-$\mathcal{B}$-positions in C, by Lemma 35, we have that $\mathcal{R}_S^* \models C\sigma'$, which implies that $R_S^* \models C\sigma$. $\square$

6 Conclusion

In this paper we have proved that well-foundedness of the term ordering is sufficient for the completeness of ordered paramodulation on Horn clauses with equality.

Our completeness proof is based on the use of $\mathcal{B}$-rewriting, a new restricted form of rewriting which forbids further rewriting in the right-hand sides introduced in the rewrite steps. Interestingly, a by-product of our results is a Knuth-Bendix completion procedure for $\mathcal{B}$-rewriting, and its completemess.

$\mathcal{B}$-rewriting is closely related to the use of *basic* strategies [BGLS95,NR95], which we believe that are fully compatible with our results (the only part of the proof that may have some problems is the so called return property). In fact, we plan to work out the basicness results using the notion of *closures* defined for clauses in [BGLS95], since it coincides, at the level of clauses, with our abstracted terms (which, by now, we only use in the proofs).

We also have to carefully explore to what extent redundancy notions are applicable in our framework for ordered paramodulation. Note that from our proof only tautology deletion is clearly possible. Subsumption seems to be possible but it may interfer with the proof of the return property. Other redundancy notions like demodulation may only be applicable in very restricted cases.

Finally we have to study how to extend our results to full first-order clauses. Here the problem comes from the fact that, within the model generation proof technique, the completeness for general clasuses is usually based on the fact that, at most one positive equation is satisfied in the generated (rewrite) model. Since, in our case, the ordering can contradict the subterm relation, we do not know how to obtain such a minimal model.

References

[BD94] Leo Bachmair and Nachum Dershowitz. Equational inference, canonical proofs, and proof orderings. *J. of the Association for Computing Machinery*, 41(2):236–276, February 1994.

[BDH86] Leo Bachmair, Nachum Dershowitz, and Jieh Hsiang. Orderings for equational proofs. In *First IEEE Symposium on Logic in Computer Science (LICS)*, pages 346–357, Cambridge, Massachusetts, USA, June 16–18, 1986. IEEE Computer Society Press.

[BG94] Leo Bachmair and Harald Ganzinger. Rewrite-based equational theorem proving with selection and simplification. *Journal of Logic and Computation*, 4(3):217–247, 1994.

[BG98] Leo Bachmair and Harald Ganzinger. Equational reasoning in saturation-based theorem proving. In W. Bibel and P. Schmitt, editors, *Automated Deduction: A Basis for Applications.* Kluwer, 1998.

[BGLS95] L. Bachmair, H. Ganzinger, Chr. Lynch, and W. Snyder. Basic paramodulation. *Information and Computation*, 121(2):172–192, 1995.

[BGNR99] Miquel Bofill, Guillem Godoy, Robert Nieuwenhuis, and Albert Rubio. Paramodulation with non-monotonic orderings. In *14th IEEE Symposium on Logic in Computer Science (LICS)*, pages 225–233, Trento, Italy, July 2–5, 1999.

[BR02] Miquel Bofill and Albert Rubio. Well-foundedness is sufficient for completeness of Ordered Paramodulation. Available at `www.lsi.upc.es/~albert/papers.html`, 2002. Long version.

[DJ90] Nachum Dershowitz and Jean-Pierre Jouannaud. Rewrite systems. In Jan van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B: Formal Models and Semantics, chapter 6, pages 244–320. Elsevier Science Publishers B.V., Amsterdam, New York, Oxford, Tokyo, 1990.

[HR91] J. Hsiang and M Rusinowitch. Proving refutational completeness of theorem proving strategies: the transfinite semantic tree method. *Journal of the ACM*, 38(3):559–587, July 1991.

[NR95] Robert Nieuwenhuis and Albert Rubio. Theorem Proving with Ordering and Equality Constrained Clauses. *Journal of Symbolic Computation*, 19(4):321–351, April 1995.

[NR01] Robert Nieuwenhuis and Albert Rubio. Paramodulation-based theorem proving. In J.A. Robinson and A. Voronkov, editors, *Handbook of Automated Reasoning*. Elsevier Science Publishers and MIT Press, 2001.

[Wec91] W. Wechler. *Universal Algebra for Computer Scientists*, volume 25 of *EATCS Monographs on Theoretical Computer Science*. Springer-Verlag, Berlin, 1991.