Asymptotic behavior and oscillations in a viscoelastic problem.

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Abstract

We propose a damped wave equation with dynamic boundary conditions as a model for a viscoelastic system. After its functional setting, we give some results on the asymptotic behavior of the solutions in three cases of special physical interest. Its behavior is strongly related with the existence of subsets of dominant eigenvalues. In the second part of this work, we consider a nonhomogeneous version of the model and we use transfer functions to approximate the behavior of its oscillating solution.

Introduction

Classically, the second order ordinary differential equation

$$m\,u''(t) + d\,u'(t) + k\,u(t) = 0 \tag{1}$$

models the dynamics of a damped spring-mass system. This consists of a spring of recovery constant k and length L, which is fixed at the end $x = 0$ and attached to a rigid mass m at the other, and where a friction force of coefficient d is damping the mass movement.

But this classical point of view neglects the possibility of dissipation due to spring internal viscosity as well as nonhomogeneous internal deformation. A more realistic approach leads us to consider a partial differential equation model, derived and justified in detail in [7] or [8], which is the following:

$$\begin{cases} \rho\,u_{tt} - E\,u_{xx} - E_1\,u_{txx} = 0, \quad 0 < x < L, \quad t > 0 \\ u(0,t) = 0 \\ m\,u_{tt}(L,t) = -E\,u_x(L,t) - E_1\,u_{tx}(L,t) - q\,u_t(L,t) \end{cases} \tag{2}$$

where $u(x,t)$ is the displacement at time t of the x particle of the spring, and ρ, E, E_1 and q are all positive parameters standing for the system density, the spring internal elasticity constant, the spring internal viscosity constant and the external damping force coefficient, respectively. Under an appropriate change of variables, functions and parameters, the equation (2) can be transformed into a dimensionless model, only depending on three parameters:

$$\begin{cases} u_{tt} - u_{xx} - \alpha\,u_{txx} = 0, \quad 0 < x < 1, \quad t > 0 \\ u(0,t) = 0 \\ u_{tt}(1,t) = -\varepsilon\,\left[u_x(1,t) + \,\alpha\,u_{tx}(1,t) + \,r\,u_t(1,t)\right] \end{cases} \tag{3}$$

We have obtained a *strongly damped linear wave equation* with a homogeneous Dirichlet boundary condition at $x = 0$ and a *dynamical boundary condition* at $x = 1$. The model parameters are now $\alpha \geq 0$, which is related to the spring internal viscosity, $r > 0$, standing for the external dissipation (due to the external damper action), and $\varepsilon > 0$, that is proportional to $1/m$. The derivation of this model can be done from a classical continuous mechanics point of view and also from a rheological approach (see [7] for details and [1] for more information about rheological models). The model with $r = 0$ and $\varepsilon = \alpha = 1$ was proposed by Grobbelaar in [3], where she used the theory of fractional powers of pairs of operators to prove the existence and uniqueness for the Cauchy problem.

Equation (3) is what we call the *dimensionless homogeneous model* and will be studied in the first part of this paper. But we can also consider an external periodic force $f(t) = A\,e^{i\omega t}$ acting onto the mass. In this case, we will work with the dimensional version of the equation which is achieved by adding this force at the $x = L$ boundary condition:

$$\left\{\begin{array}{l} \rho\, u_{tt} \;=\; E_1\, u_{txx} + E\, u_{xx}, \quad 0 < x < L,\ t > 0 \\ u(0,t) = 0 \\ m u_{tt}(L,t) = -E u_x(L,t) - E_1 u_{tx}(L,t) - q u_t(L,t) + A e^{i\omega t} \end{array}\right. \tag{4}$$

This is what we call the *nonhomogeneous model* and its study is the aim of the second part of this work.

As we have introduced the partial differential equation model instead of the classical ordinary differential one, we are interested in comparing these two approaches to the same system. We expect that, at least in some cases, the partial differential equation solution will behave as an ordinary differential equation one for large times. This asymptotic behavior will be studied in the case of the homogeneous model (3) by looking for the existence of *dominant eigenvalues* of the operator. The functional framework for the problem will be given in the next section of this paper. In the last section, the oscillating solutions of the dimensional nonhomogeneous model (4) will be considered and, using transfer functions, we will approximate them by the oscillating solutions of some ordinary differential equations in a sense that will be explained.

The homogeneous model: main results.

In this section we introduce the general functional setting and outline the main results for the linear homogeneous problem (for more details, see [7] and [8]).

Let us begin with the definition of the operator and the spaces, based on [3]. We consider the following spaces:

$$X_2 = \{(u,\gamma) \in H^2(0,1) \times \mathbb{C},\ u(1) = \gamma,\, u(0) = 0\} \ \text{ as a subspace of } H^2(0,1) \times \mathbb{C}\ ;$$

$$X_1 = \{(u,\gamma) \in H^1(0,1) \times \mathbb{C},\ u(1) = \gamma,\, u(0) = 0\} \ \text{ as a subspace of } H^1(0,1) \times \mathbb{C}\ ;$$

$$X_0 = \{(u,\gamma) \in L^2(0,1) \times \mathbb{C}\}, \quad \text{which is equal to } L^2(0,1) \times \mathbb{C}.$$

These are Hilbert spaces with the inner products:

$$\langle (u,u(1)),(v,v(1))\rangle_{X_1} = \int_0^1 u_x\, \overline{v_x}\, dx \quad \text{and} \quad \langle (u,\gamma),(v,\beta)\rangle_{X_0} = \int_0^1 u\, \overline{v}\, dx \;+\; \frac{1}{\varepsilon}\gamma\,\overline{\beta}$$

which are equivalents to the natural ones (see [8]). The domain of the operator A_α is then defined as:

$$\mathcal{D}(A_\alpha) = \left\{ \begin{pmatrix} (u, u(1)) \\ (v, v(1)) \end{pmatrix} \in X_1 \times X_1,\ (u + \alpha\, v) \in H^2(0,1) \right\} \subset \mathcal{H}$$

where $\mathcal{H} = X_1 \times X_0$ is a Hilbert space with the corresponding inner product.

Remark 0.1 *In the special case of $\alpha = 0$, the domain turns to be $\mathcal{D}(A_0) = X_2 \times X_1$, which is included in the previous general definition.*

The Cauchy problem coming from equation (3) can be now written as the evolution equation:

$$\begin{cases} \frac{d}{dt} V = A_\alpha V, & t \in (0, \infty) \\ V(0) = F_0 \end{cases} \tag{5}$$

where, if $V = \begin{pmatrix} (u, u(1)) \\ (v, v(1)) \end{pmatrix} \in \mathcal{D}(A_\alpha)$ then

$$A_\alpha V = \begin{pmatrix} (v, v(1)) \\ (\,(u + \alpha\, v)_{xx}, -\varepsilon\,(u + \alpha\, v)_x(1)\ -\ \varepsilon\, r\, v(1)\,) \end{pmatrix}$$

and F_0 stands for the initial conditions. The details of the next theorem, which follows [3] and [6], are given in [8].

Theorem 0.2 *If $\alpha > 0$, the operator $(A_\alpha, \mathcal{D}(A_\alpha))$ is the infinitesimal generator of an analytic semigroup in $\mathcal{H}$. If $\alpha = 0$, the operator $(A_0, \mathcal{D}(A_0))$ is the generator of a C_0 contraction semigroup in $\mathcal{H}$. Then, existence and uniqueness of solutions for (5) are guaranteed for $\alpha \geq 0$.*

With the functional setting given, we want now to study the model (3) from a parametric point of view. Our main interest is studying the existence of a limit ordinary differential equation in three special cases: for a purely elastic spring (that is $\alpha = 0$) we can conclude the nonexistence of such a limit equation, and we prove that all solutions tend to zero as time tends to ∞ and that there exist those which tend to zero as slow as we wish (see [8]); for models with small internal viscosity (α near 0), the existence of the limit equation is proved, but its order as well as its coefficients cannot be found for a general α; the case of a large mass at the end (or ε near 0), also of physical interest, is one where this limit ordinary differential equation can be analytically approximated and, if the first coefficient is taken equal to m in order to compare the resulting equation with the classical one, the limit equation is:

$$m\, w''(t) + k_1\, w'(t) + k_0\, w(t) = 0 \tag{6}$$

where

$$k_1 = \left(\frac{E_1}{L} + q\right) - \frac{1}{3}\left(\frac{E_1}{L} + q\right)\left(\frac{\rho L}{m}\right) + \left(\frac{4\,E_1}{45\,L} + \frac{q}{15}\right)\left(\frac{\rho L}{m}\right)^2 + \ldots$$

$$k_0 = \frac{E}{L}\left[1 - \frac{1}{3}\left(\frac{\rho L}{m}\right) + \frac{4}{45}\left(\frac{\rho L}{m}\right)^2 + \left(\frac{q^2}{45\,E\rho} - \frac{16}{945}\right)\left(\frac{\rho L}{m}\right)^3 + \ldots\right]$$

If we compare this equation with the classical spring-mass equation given in (1), we can observe that we have obtained more accurate coefficients: the damping one includes all the system damping parameters and the one related with the recovery force of the spring involves system elasticity, but also other system parameters such as, for instance, its density and length.

The specific sense of this ordinary differential equation as a limit of the model (3) is given by the dominant eigenvalues, which are introduced following the ideas given in [4]. The main idea is that if we have a finite subset of eigenvalues whose real part is strictly greater than the real part of rest of the operator spectrum (it is called a finite subset of dominant eigenvalues), the solution of (3) tends to behave as the solution of the equation generated by this finite subset, so as a finite dimensional equation solution, which is treated as a limit of the partial differential equation model.

The results obtained in this section can be seen in more detail in [8] and are proved using the spectral and perturbation theory of operators (see [2] and [5]).

Non-homogeneous model and transfer functions.

We are now interested in the nonhomogeneous model (4), where an external periodic force of amplitude A and frequency ω, that is $f(t) = A\, e^{i\omega t}$, has been applied onto the mass. In this section, we will look for solutions that are globally bounded in time, that is solutions that are bounded for all $t \in (-\infty, \infty)$, and a new concept of approximative ordinary differential equation in terms on its own globally bounded solution will be given.

Let us begin with a brief review of a well known theory for the second order non-homogeneous ordinary differential equation

$$k_2\, z''(t) + k_1\, z'(t) + k_0\, z(t) = A\, e^{i\omega t} \tag{7}$$

that has a unique globally bounded solution when $k_0,\ k_1,\ k_2 > 0$, as the unique globally bounded solution of the corresponding homogeneous equation is $z(t) \equiv 0$. If we look for a solution of (7) of the form $z_b(t) = B\, e^{i\omega t}$, we find the solution that is globally bounded in time:

$$z_b(t) = A\, H_{ode}(\omega)\, e^{i\omega t} \tag{8}$$

where

$$H_{ode}(\omega) = \frac{1}{k_0 - \omega^2\, k_2 + k_1\, i\, \omega} \tag{9}$$

Definition 0.3 *The ω-function $H_{ode}(\omega)$ is called the* transfer function *of the second order ordinary differential equation (7).*

The behavior of $H_{ode}(\omega)$, which depends on the value of the critical parameter k_1, can be seen in figure (1). We observe, for instance, the resonance phenomenon for $k_1 = 0$.

If we now look for solutions of the partial differential equation model (4) which are globally bounded in time, we have the following result.

Lemma 0.4 *Equation (4) has a unique solution globally bounded in time, which is:*

$$u_b(x,t) = A\, H_{pde}(x, \omega)\, e^{i\omega t} \tag{10}$$

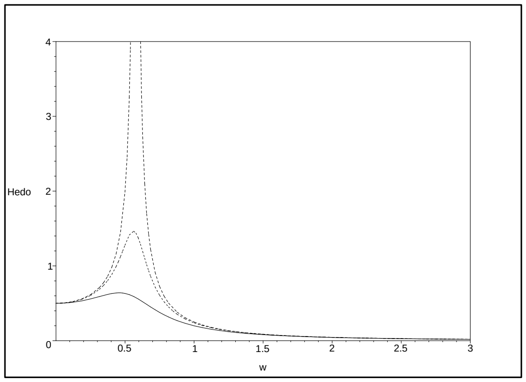

Figure 1: *Transfer function module of the ODE (7) when $k_1 = 0$, k_1 is small and k_1 is large, respectively.*

where

$$H_{pde}(x,\omega) = \frac{\sinh(ax)}{[-\omega^2 m \sinh(aL) + Ea\cosh(aL)] + i\omega[E_1 a\cosh(aL) + q\sinh(aL)]} \tag{11}$$

and $a^2 = \dfrac{-\omega^2 \rho}{E + E_1 \omega i}$

Proof. The only globally-bounded-in-time solution of (4) when $f(t) = 0$ is $u(x,t) \equiv 0$ (this is proved in [8] using semigroup properties of equation (3)), so the nonhomogeneous equation (4) has a unique solution globally bounded in time. To find it, it suffices to look for a solution of (4) of the form $u_b(x,t) = B\, e^{i\omega t} \sinh(ax)$. □

Definition 0.5 *The function $H_{pde}(x,\omega)$ given in (11) is called the* transfer function *at the point x of problem (4).*

As we have done for the ordinary differential equation transfer function, we can draw a graphic of $H_{pde}(x,\omega)$ for different values of the critical parameters, which now are E_1 and q, and a fixed point, take for instance $x = L$. This is done in figure (2), where we can observe the higher presence of resonance zones for low values of those dissipative parameters.

It can be seen in the figure (2) that the graphics of $H_{pde}(x,\omega)$ are quite similar to the $H_{ode}(\omega)$ ones when ω is near zero (remember figure (1)). This gives us an idea of what to do next: as the globally bounded solutions $u_b(x,t)$ and $z_b(t)$ are essentially given by these transfer functions, it is reasonable, at least for low frequencies, to find the coefficients of equation (7) such that its associated transfer function is as similar as possible to $H_{pde}(x,\omega)$ for a given x (say $x = L$) when ω is near zero. This is achieved if the power series expansions around $\omega = 0$ have as many equal terms as possible.

As the power series expansion of $H_{ode}(\omega)$ only involves three unknown parameters (k_0, k_1 and k_2), it will be enough expanding $H_{ode}(\omega)$ and $H_{pde}(L,\omega)$ up to order 3:

$$H_{ode}(\omega) = \frac{1}{k_0} - i\,\frac{k_1}{(k_0)^2}\,\omega + \frac{\frac{k_2}{k_0} - \frac{k_1^2}{(k_0)^2}}{k_0}\,\omega^2 + o\left(\omega^3\right) \tag{12}$$

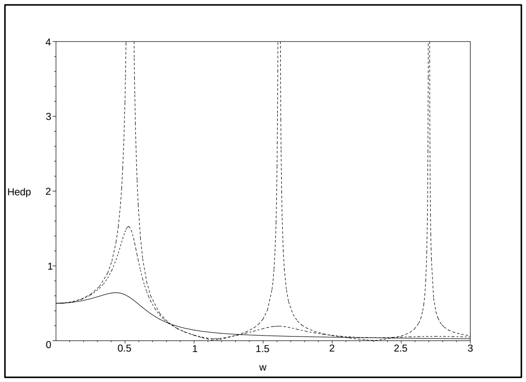

Figure 2: *Module of the transfer function for the PDE model (4) when $E_1, q = 0$, E_1, q are small and E_1, q are large, respectively.*

and

$$H_{pde}(L,\omega) \; = \frac{L}{E} - i\,\frac{L(E_1+qL)}{E^2}\,\omega - \frac{L\left(-3LmE - L^2E\rho + 3E_1^2 + 6E_1qL + 3q^2L^2\right)}{3E^3}\,\omega^2 + o(\omega^3) \tag{13}$$

Equaling the first three terms of (12) to the first three of (13) we obtain:

$$k_0 = \frac{E}{L},\; k_1 = \frac{E_1}{L} + q,\; k_2 = m + \frac{1}{3}\rho L$$

So, the approximative second order ordinary differential equation turns to be:

$$\left(m + \frac{1}{3}\rho L\right)\, z''(t) + \left(\frac{E_1}{L} + q\right)\, z'(t) + \left(\frac{E}{L}\right)\, z(t) = A\, e^{iwt} \tag{14}$$

In this case, it can be seen that the approximation is even better than expected, because $|H_{edo}(\omega) - H_{edp}(L,\omega)| = o(\omega^4)$, so $|u_b(L,t) - z_b(t)| = A\, o(\omega^4)$. We can also compare the moduli of both transfer functions, as it is done in the figures (3) and (4), for different values of E_1 and q. It can be seen that near $\omega = 0$ they are really very close. In fact, the problems arise in those resonance parts of the graph, which are more present for small values of the dissipation parameters E_1 and q.

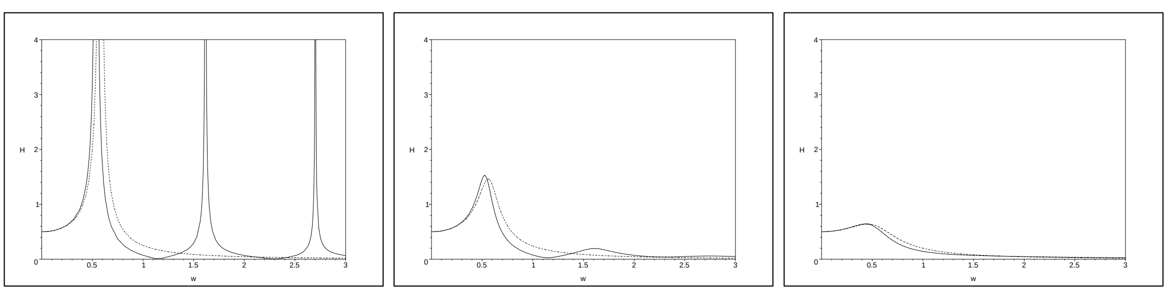

Figure 3: *Moduli of $H_{pde}(L,\omega)$ and the obtained $H_{ode}(\omega)$ (in dots) when $E_1, q = 0$, E_1, q are small and E_1, q large, respectively.*

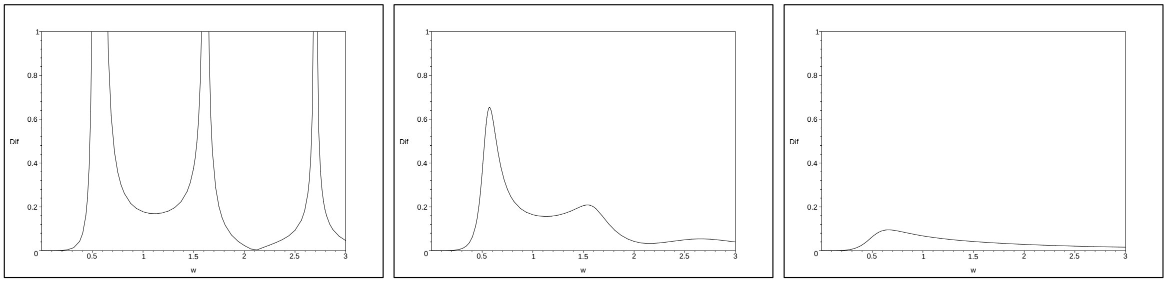

Figure 4: *The modulus of the difference between transfer functions of figure (3) for the same values of the parameters.*

We can now compare equation (14) with equation (6), the one obtained in the previous section using dominant eigenvalues theory. In the case of using dominant eigenvalues, a better limit equation means taking a better approximation of the coefficients of the ordinary differential equation, always of second order, and this is achieved with an improved approximation of the two dominant eigenvalues. Furthermore, equation (6) gives an asymptotic behavior of any solution of the homogeneous partial differential equation model, so it can be a good approximation for the partial differential equation problem if time is large enough. But in this section, the aim has been to find an ordinary differential equation whose globally bounded solution behaves, for each time and at a given point, as the only globally bounded one of (4). This approximative equation has exact coefficients and to improve it it is necessary to increase the number of terms between transfer functions expansions which are equal. But the only way to do this is to consider a higher order ordinary differential equation, as it will have more free coefficients. This can be done in the same way as in order 2 one, because a n-order differential equation has also an associated transfer function, which will be expanded up to order n (see [8]).

Finally, the motions of the spring-mass system acting under the globally bounded solution of the partial differential equation as well as the obtained ordinary differential equation (14) can be simulated and compared for different times, using the analytical expression for $u_b(L,t)$ and $z_b(t)$. In the figure (5), this comparison is done when acting a low frequency external force and it can be seen the similitude between both approximations.

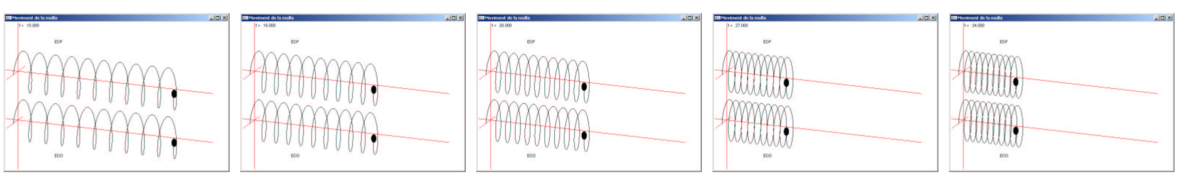

Figure 5: *Simulation of the PDE and second order ODE (14) globally bounded solution for different times when* $f(t) = 5e^{0.1\,i\,t}$.

For the contrary, we can see in the figure (6) the different behavior of both points of view when acting a higher frequency external force. It is also clear the nonhomogeneous internal deformation in the case of the partial differential equation spring.

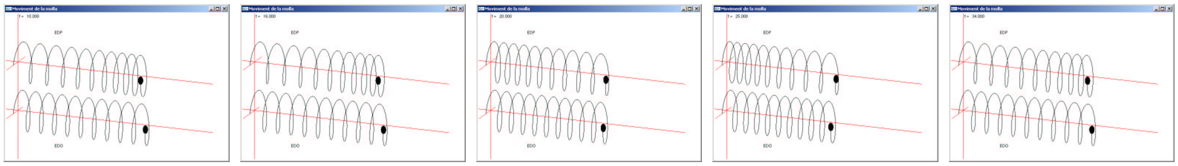

Figure 6: *Simulation of the PDE and second order ODE (14) globally bounded solution for different times when* $f(t) = 7e^{it}$.

Once again, it arises the need of both approaches: while the ordinary differential equation one gives us a simplified first approach to the globally bounded system behavior, we have seen different situations where it is not always enough to explain some phenomenons which clearly appear for some values of the system parameters.

Acknowledgments

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