Spectral analysis and large time behaviour in a viscoelastic model

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Outline

- The problem.
  - a spring-mass-damper system: motivation and modelling;
  - main tools: dominant eigenvalues;
  - functional setting.

- Main results: spectral analysis and large time behaviour.
  - large mass at the end (ε small);
  - the role of spring internal viscosity (α).

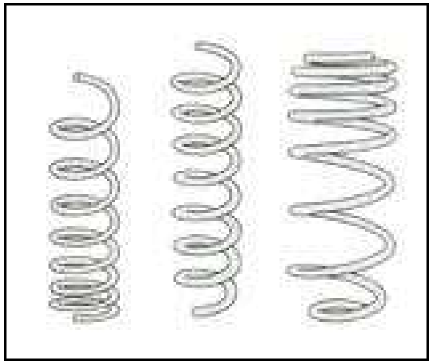
THE PROBLEM

- A spring-mass-damper system: motivation and modelling.
- Main tools: dominant eigenvalues.
- Functional setting.

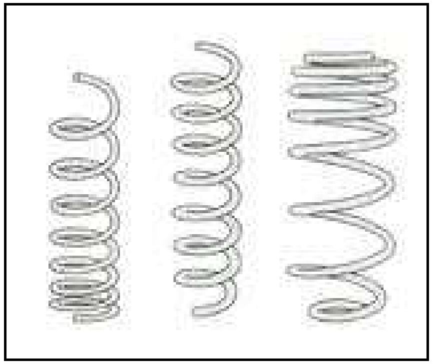
The system

A damped spring-mass system

Present in: automotion, seismic control, ...



Spring or **viscoelastic** device
(elasticity + possible **internal** dissipation)
fixed at one end



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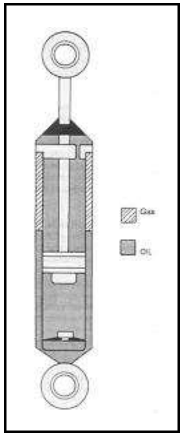
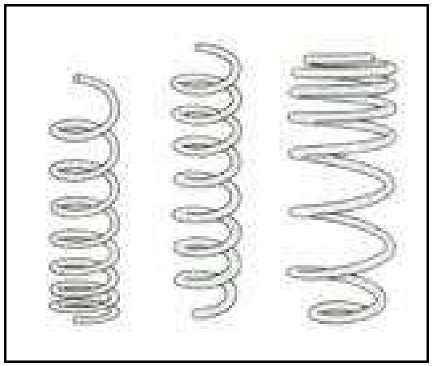


Damper or **viscous** device
(**external** dissipation)
acting onto the mass at the other end



Spring or **viscoelastic** device
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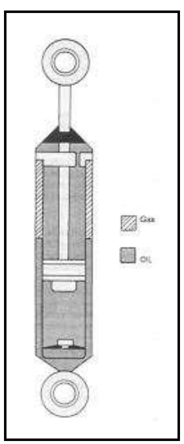
$\Rightarrow$ Equation



Damper or **viscous** device
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acting onto the mass at the other end

$\Rightarrow$ Boundary conditions

Motivating the problem: PDE vs. ODE

Spring-mass system as an ODE (spring as a discrete system):

$$m\,u''(t) + r\,u'(t) + k\,u(t) = 0$$

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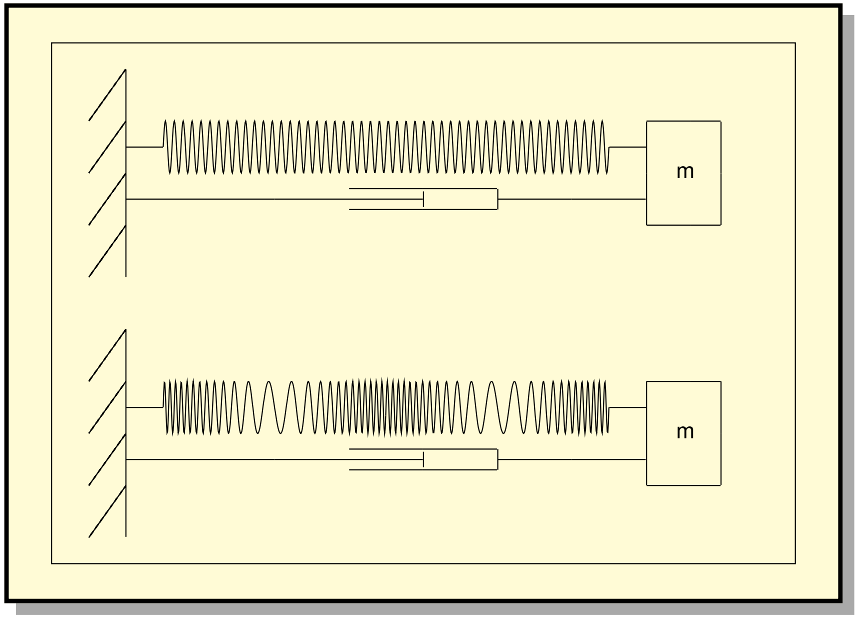
But ...



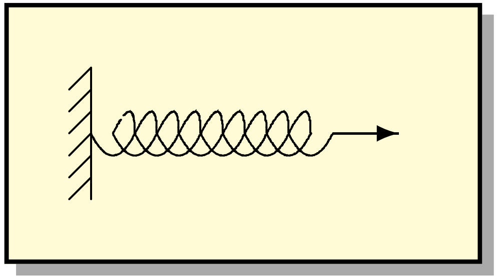
- differences in internal deformation?
- possible internal dissipation?

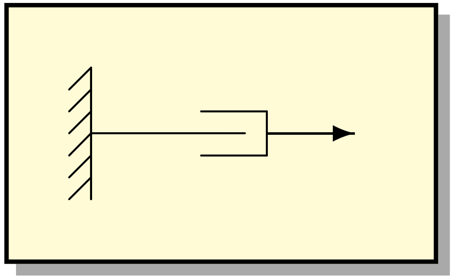
Modelling

- Momentum balance law
- **Rheological** approach

spring (elasticity) and **dashpot** (viscosity)
combined in series or parallel

$$\sigma_e = E\,\varepsilon_e \qquad \sigma_v = E_1\,\dot{\varepsilon}_v$$





Modelling

- Momentum balance law ✓
- **Rheological** approach ✓

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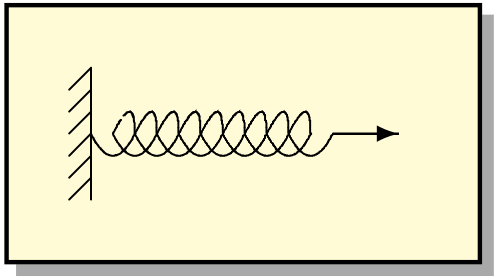
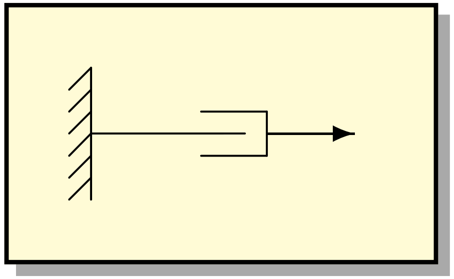


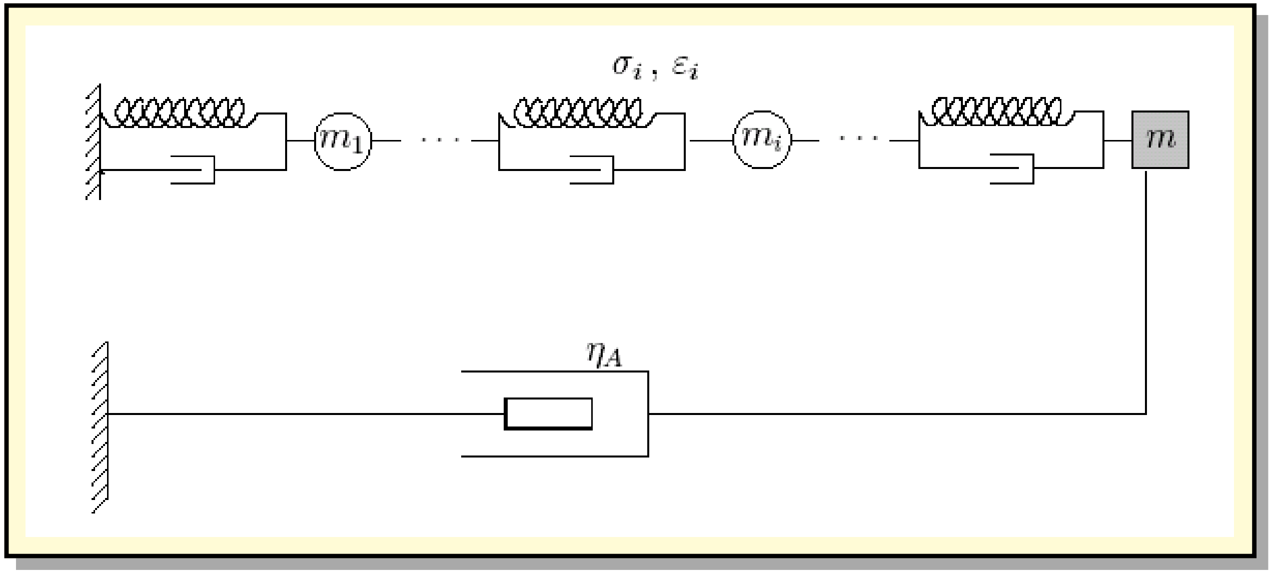


Continuous Kelvin-Voigt model

or

wave equation with strong damping

(our case)



Rheological model:

E, E_1 *spring elasticity and viscosity;* q *external viscosity.*

Wave equation with **strong damping** and **dynamical** boundary conditions (linear and dimensionless model)

$u(x,t)$ = displacement of the x particle at time t

$$\begin{cases} u_{tt} - u_{xx} - \alpha\, u_{txx} = 0, \quad 0 < x < 1,\ t > 0 \\ u(0,t) = 0 \\ u_{tt}(1,t) = -\varepsilon\ \left[u_x(1,t) + \alpha\, u_{tx}(1,t) + r\, u_t(1,t)\right] \end{cases}$$

$\alpha \geq 0 \rightsquigarrow$ internal damping (spring, E_1) [a]

$r > 0 \rightsquigarrow$ external dissipation (external damper, q)

$\varepsilon \geq 0 \rightsquigarrow$ inverse of the external mass [b]

$(\alpha = \frac{E_1}{\sqrt{E\rho}L}, r = \frac{q}{\sqrt{E\rho}}, \varepsilon = \frac{\rho L}{m})$

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[a] ***"Spectral analysis and limit behaviors in a spring-mass system"***, MP, J. Sola-Morales. Preprint (2005)

[b] ***"Analysis of a viscoelastic spring-mass model"***, MP, J. Sola-Morales. JMAA (2004)

Wave equation with **strong damping** and **dynamical** boundary conditions (linear and dimensionless model)

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- ***M. Grobbelaar-van Dalsen (1994):*** $\alpha = \varepsilon = 1$ **,** $r = 0$ *; P. Massat (1983).* ***S.Chen, K.Liu and Z.Liu (1999)*** *(independently).*
- ***D.L. Russell (1986):*** *explicit case* $\varepsilon = 0$ *for elastic beam equation.*

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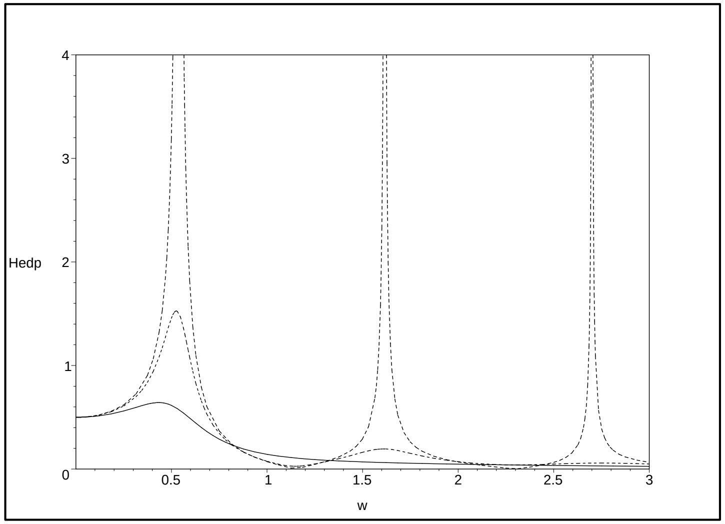
- ***Elastic medium + rigid mass:*** *C.Castro, E.Zuazua '98 (controllability) ; ...*
- ***Wave eq. with strong damping:***
  N. Cónsul, J. Solà-Morales '99 (equilibria stability) ;
  P. Freitas'97 (spectral analysis and stability);
- ***Wave eq. with weak damping + dynamical b.c.:***
  A. Freiria Neves, H. de Souza Ribeiro, O. Lopes '86.
- *...*

Non-homogeneous version [a]

$$\begin{cases} u_{tt} - u_{xx} - \alpha\, u_{txx} = 0, \quad 0 < x < 1,\ t > 0 \\ u(0,t) = 0 \\ u_{tt}(1,t) = -\varepsilon\,[u_x(1,t) + \alpha\, u_{tx}(1,t) + r\, u_t(1,t)] + f(t) \end{cases}$$

globally bounded solutions $\rightsquigarrow$ *resonances*



[a] *Asymptotic behaviour and oscillations in a viscoelastic problem*, MP, Proceedings XVIII CEDYA-VIII CMA (2003).

Nonlinear problem [a]

$$\begin{cases} u_{tt} - u_{xx} - \alpha\, u_{txx} + \varepsilon f\left(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}\right) = 0, \quad 0 < x < 1,\ t > 0 \\ u(0,t) = 0 \\ u_{tt}(1,t) = -\varepsilon\, [u_x(1,t) + \alpha\, u_{tx}(1,t) + r\, u_t(1,t)] - \varepsilon f\left(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}\right) \end{cases}$$

feedback control, exponentially attracting invariant manifold

(A.N. Carvalho, 1995; A.N. Carvalho, G. Lozada-Cruz, 2001)

[a] *In preparation, MP.*

Main tools

spring-mass-damper system

$\Downarrow$

PDE model or ODE model? $\Rightarrow$ comparison between models

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spring-mass-damper system

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PDE model or ODE model? $\Rightarrow$ comparison between models

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Large time behaviour of solutions: reduction to ODEs

Main tools

spring-mass-damper system

$\Downarrow$

PDE model or ODE model? $\Rightarrow$ comparison between models

$\Downarrow$

Large time behaviour of solutions: reduction to ODEs $\checkmark$

Dominant eigenvalues $\checkmark$

(generalized convergence of operators, characteristic equation)

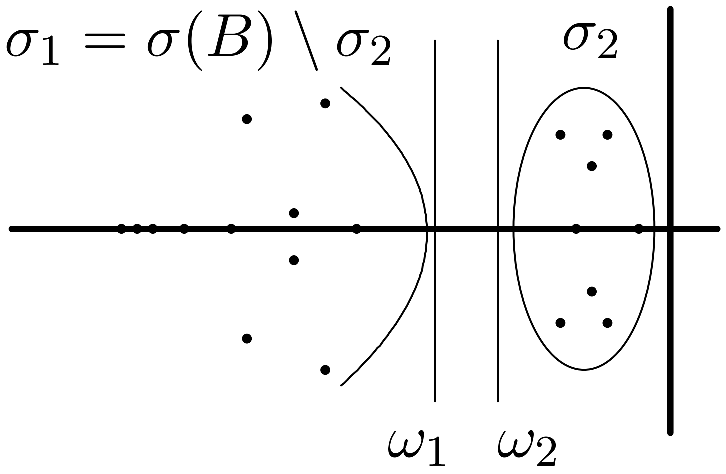
1. Dominant eigenvalues.

Take the evolution equation $\frac{d}{dt}x(t) = B\,x(t)$ with B analytic.

Def: We call σ_2 a set of **dominant eigenvalues of** B if

$\sigma(B) = \sigma_1 \cup \sigma_2,$ where σ_2 satisfies:

- finite subset of isolated eigenvalues
- finite algebraic multiplicity
- $Re\,(\sigma_1) < \omega_1 < \omega_2 < Re\,(\sigma_2)$ (if $Re\,(\sigma_2) = C$, we call σ_2 the **maximal dominant** set).



We then have a natural decomposition of the spectrum:

$$\sigma(B) = \sigma_1 \cup \sigma_2,$$

of the total space: $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$
and of the operator: $B = (B_1, B_2)$.

Thm: In this case, the equation of finite dimension

$$\frac{d}{dt} x_2 = B_2 x_2$$

can be thought as the **limit** when $t \to +\infty$ of $\frac{d}{dt} x = Bx$.

(***Proof:*** $\lim_{t \to \infty} \frac{\|x(t) - x_2(t)\|}{\|x(t)\|} = 0$)

2. Generalized convergence of operators (T. Kato).

Def: Let T_n, $n \in \mathbb{N}$ and T be closed operators. We say

$$T_n \underset{n\to\infty}{\longrightarrow} T \quad \textbf{in the generalized sense} \text{ if } \boxed{\hat{\delta}(T_n, T) \underset{n\to\infty}{\longrightarrow} 0}$$

where $\hat{\delta}$ is measures the *gap* between the graphs.

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Generalization of norm-convergence:

- gen. convergence $\Leftrightarrow \|T_n - T\| \to 0$ if bounded.
- gen. convergence $\Leftrightarrow \|T_n^{-1} - T^{-1}\| \to 0$ if bounded inverses.
- ...

Thm [T. Kato] (Semicontinuity of separated parts of the spectrum):

Let T closed and let $\sigma(T)$ **separated** in two parts by a closed curve Γ. For every closed operator S s.t. $\hat{\delta}(T,S)$ is small enough, we have $\sigma(S)$ **separated in the same way.**

(isomorphic eigenspaces, norm convergence of projections, ...)

$$\Downarrow$$

continuity of finite sets of eigenvalues ...

3. The characteristic equation

By considering solutions of the form

$$u(x,t) = e^{\lambda t} u(x)$$

it can be easily seen that the eigenvalues λ of $A_\alpha(\varepsilon)$ are the roots of the following characteristic equation:

$$e^{\frac{2\lambda}{\sqrt{1+\lambda\alpha}}} = \frac{\lambda - \varepsilon\sqrt{1+\lambda\alpha} + \varepsilon\, r}{\lambda + \varepsilon\sqrt{1+\lambda\alpha} + \varepsilon\, r}$$

Functional setting

$$X_2 = \{(u,\gamma) \in H^2(0,1) \times \mathbb{C},\ u(1) = \gamma,\ u(0) = 0\} \subset H^2(0,1) \times \mathbb{C}$$

$$X_1 = \{(u,\gamma) \in H^1(0,1) \times \mathbb{C},\ u(1) = \gamma,\ u(0) = 0\} \subset H^1(0,1) \times \mathbb{C}$$

$$X_0 = \{(u,\gamma) \in L^2(0,1) \times \mathbb{C}\} = L^2(0,1) \times \mathbb{C}$$

We can write the lineal model as the following evolution equation:

$$\begin{cases} \frac{d}{dt}V - A_\alpha(\varepsilon)V = 0, \quad t \in (0,\infty) \\ V(0) = F_0 \end{cases}$$

where

$$A_\alpha(\varepsilon)\begin{pmatrix}(u,u(1))\\(v,v(1))\end{pmatrix} = \begin{pmatrix}(v,v(1))\\ (\,(u+\alpha\,v)_{xx}, -\varepsilon\,(u+\alpha\,v)_x(1) - \varepsilon\,rv(1)]\,)\end{pmatrix}$$

$$\mathcal{D}(A_\alpha(\varepsilon)) = \left\{\begin{pmatrix}(u,u(1))\\(v,v(1))\end{pmatrix} \in X_1 \times X_1,\ (u+\alpha\,v) \in H^2(0,1)\right\}$$

$$\subset\ \mathcal{H} = X_1 \times X_0$$

Thm:

- $(A_\alpha, \mathcal{D}(A_\alpha))$ with $\alpha > 0$ generates an analytic semigroup in $\mathcal{H}$ [*using P.Massat, S.Chen- K.Liu- Z.Liu*].
- $(A_0, \mathcal{D}(A_0))$ generates a $\mathcal{C}^0$ semigroup of contractions in $\mathcal{H}$.

MAIN RESULTS

- Large mass at the end (ε small).
- The role of spring internal viscosity (α).

Main results

We have written the PDE model as an evolution equation:

$$\frac{d}{dt}V - A_\alpha(\varepsilon)V = 0$$

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$$\frac{d}{dt}V - A_\alpha(\varepsilon)V = 0$$

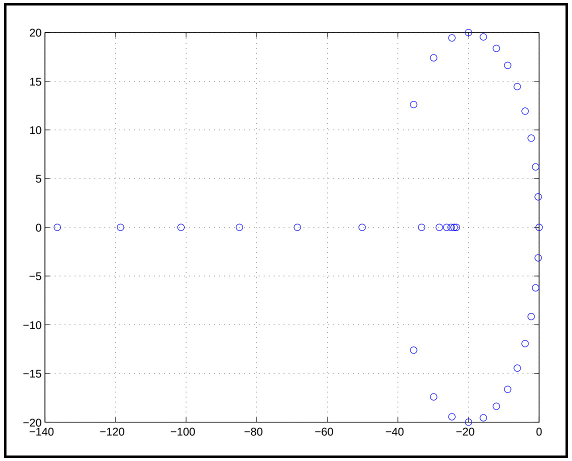
$$\Downarrow$$

PDE $\longrightarrow$ classical ODE ? $\Leftrightarrow$ dominant eigenvalues?

Two points of view (**physical** and **mathematical** interest):

- ε small (large mass at the end).
- α (internal viscosity):
    - $\alpha = 0$ or purely elastic spring;
    - $\alpha \sim 0$ or small internal viscosity;
    - $\alpha >> 1$: the overdamping case.

1a. Infinite mass at the end or $\varepsilon = 0$ $(\alpha, r > 0)$.



(explicit eigenvalues)

- $\lambda_0 = 0$ is a **double** dominant eigenvalue.

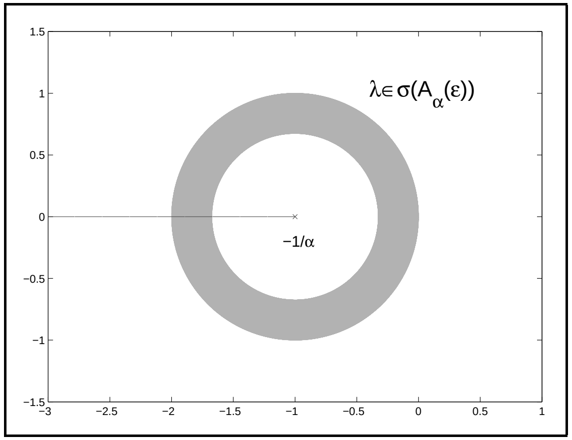
*(**Proof:** Energies)*

- $\sigma_{ess}(A_\alpha(0)) = \{-1/\alpha\}$.

*(**Proof:** Relatively compact perturbation of an operator with explicit computable σ_{ess})*

1b. Large mass at the end or small $\varepsilon > 0$ ($\alpha, r > 0$).

- **Thm:** Spectrum as a perturbation of the limit case one (when $\varepsilon\alpha r < 1$).



*(**Proof:** Analysis of the characteristic equation)*

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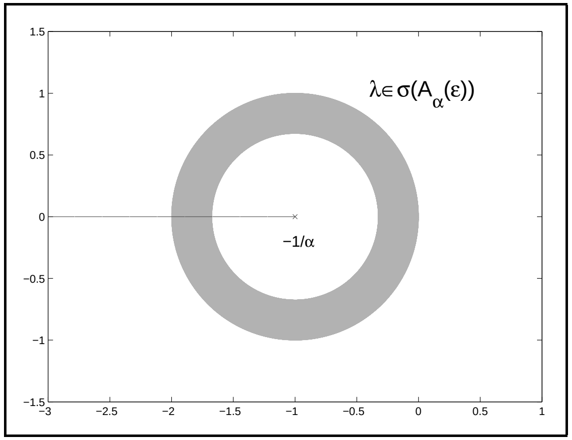


*(**Proof:** Analysis of the characteristic equation)*

- **Thm:** Existence of **2 dominant eigenvalues** $\{\lambda_0^+(\varepsilon), \lambda_0^-(\varepsilon)\}$ if ε small enough (explicit perturb. of $\lambda(0) = 0$)

*(**Proof:** $A_\alpha(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} A_\alpha(0)$ in a generalized sense, ...)*

Thm (limit ODE):

For m sufficiently large, the **asymptotic behaviour** of the PDE solutions at $x = L$ is given by the solutions of the following 2nd order ODE:

$$m\,w'' + k_1\,w' + k_0\,w = 0 \quad \textit{(dimensional)}$$

where $w(t) = u(L,t)$ and

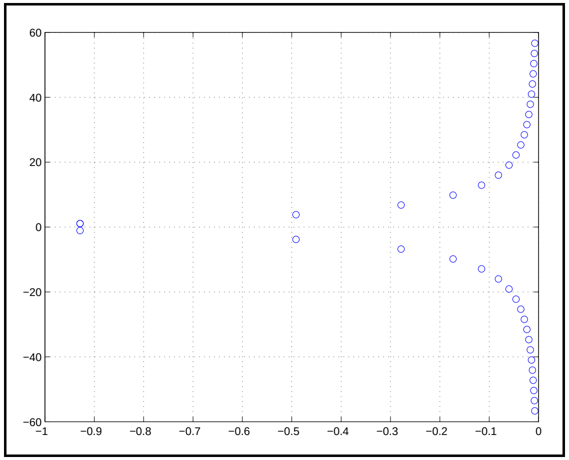
$$k_1 = \left(\frac{E_1}{L} + q\right) - \frac{1}{3}\left(\frac{E_1}{L} + q\right)\left(\frac{\rho L}{m}\right) + \left(\frac{4\,E_1}{45\,L} + \frac{q}{15}\right)\left(\frac{\rho L}{m}\right)^2 + \dots$$

$$k_0 = \frac{E}{L}\left[1 - \frac{1}{3}\left(\frac{\rho L}{m}\right) + \frac{4}{45}\left(\frac{\rho L}{m}\right)^2 + \left(\frac{q^2}{45\,E\rho} - \frac{16}{945}\right)\left(\frac{\rho L}{m}\right)^3 + \dots\right]$$

PDE $\longrightarrow$ ODE, but observe the coefficients !!

2a. Purely elastic spring or $\alpha = 0$ ($\varepsilon, r > 0$)

- A_0 generator of a $\mathcal{C}^0$ semigroup of contractions.
- Spectrum:
  - $\sigma(A_0) = \sigma_p(A_0)$ (not essential spectrum)
  - infinitely many, isolated, finite algebraic multiplicity.
  - $Re\,\lambda < 0$ for all $\lambda \in \sigma(A_0)$.
  - $Re\,\lambda_n \to 0$ and $Im\,|\lambda_n| \to \infty$.



2a. Purely elastic spring or $\alpha = 0$ ($\varepsilon, r > 0$)

- A_0 generator of a $\mathcal{C}^0$ semigroup of contractions.

*(**Proof:** $-A_0$ maximal monotone operator)*

- Spectrum:
  - $\sigma(A_0) = \sigma_p(A_0)$ (not essential spectrum)
  - infinitely many, isolated, finite algebraic multiplicity.
  - $Re\,\lambda < 0$ for all $\lambda \in \sigma(A_0)$.
  - $Re\,\lambda_n \to 0$ and $Im\,|\lambda_n| \to \infty$.

*(**Proof:** Analysis of eigenfunctions,*
characteristic equation for eigenvalues and energy)

Thm (Asymptotic behaviour of solutions):

- Not existence of any finite subset of dominant eigenvalues:

  PDE $\nrightarrow$ ODE!!

- But ... all solutions **tend to 0** when $t \to +\infty$.

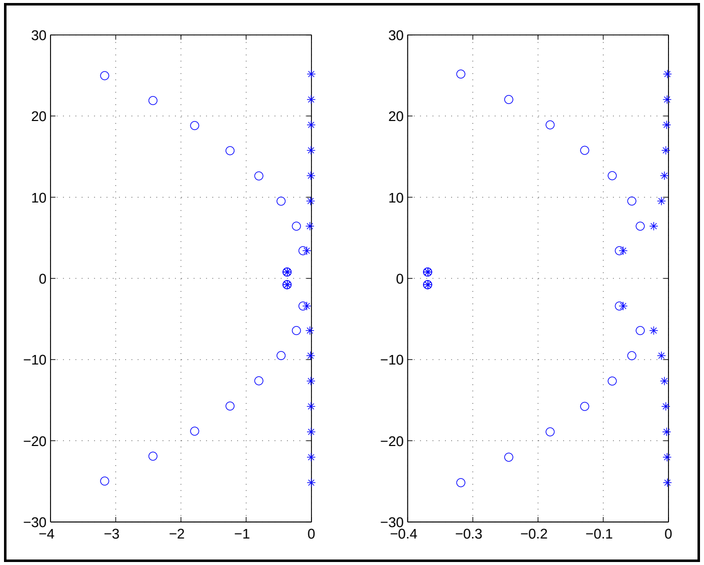
  And there exist solutions which tend to zero **as slow as we wish** (because no internal damping).

  Polinomial decay for smoother solutions?

2b. Viscoelastic spring with small $\alpha > 0$ ($\varepsilon, r > 0$)

- A_α is the generator of an analytic semigroup if $\alpha > 0$ (and $\varepsilon \geq 0$).

- Spectrum:
  - $\sigma_{ess}(A_\alpha) = \{-1/\alpha\}$, $\alpha > 0$;
  - $\sigma_p(A_\alpha) \underset{\alpha \to 0}{\longrightarrow} \sigma_p(A_0)$ in compact sets, but **not globally**.



$(\alpha = 0.01)$ $(\alpha = 0.001)$

2b. Viscoelastic spring with small $\alpha > 0$ ($\varepsilon, r > 0$)

- A_α is the generator of an analytic semigroup if $\alpha > 0$ (and $\varepsilon \geq 0$).

  *(**Proof:** using Massat'83)*

- Spectrum:
  - $\sigma_{ess}(A_\alpha) = \{-1/\alpha\}$, $\alpha > 0$;
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  *(**Proof:** $A_\alpha \underset{\alpha \to 0}{\longrightarrow} A_0$ in a generalized sense,*
  *but A_α is **sectorial** when $\alpha > 0$)*

Thm (Existence of dominant eigenvalues and limit ODE):

If $\alpha > 0$ is small enough,

there **exists** a finite subset of dominant eigenvalues, but ...

$$\Downarrow$$

PDE $\longrightarrow$ ODE ?

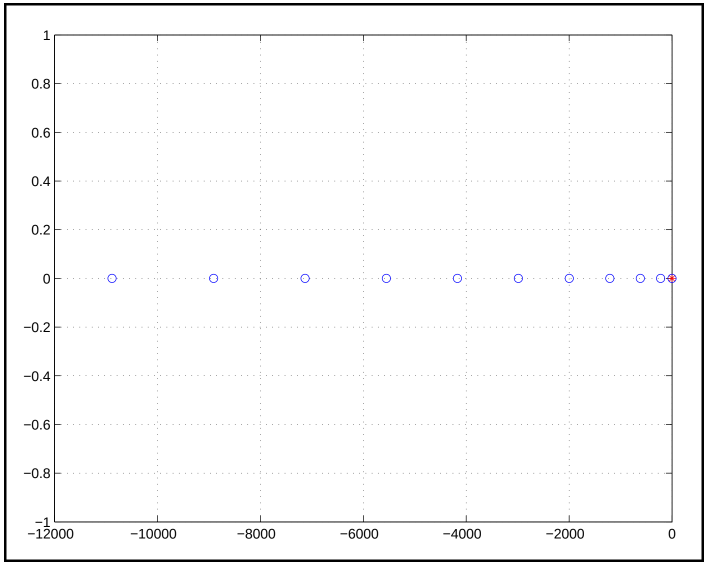
$$\Downarrow$$

Yes, but the **order** and **coefficients** of the ODE may be differents for each α.

(the limit ODE may **not** be of order 2 !!)

2c. Overdamped viscoelastic spring or $\alpha > 0$ large

- Explicit limit case when $\varepsilon = \infty$, $r = 0$:



If α large enough:

- $\sigma_p = \{\lambda_n^{\pm}\} \subset \left(-\infty, -\frac{1}{\alpha}\right)$
- $\lambda_n^- \to -\infty$, $\lambda_n^- \to -\frac{1}{\alpha}$
- $\sigma_{ess} = \left\{-\frac{1}{\alpha}\right\}$

Overdamping for α large.

- If ε large, r small and α **large enough** ...

  Thm (No limit ODE):

  We have $\sigma \subset \left(-\infty, -\frac{1}{\alpha}\right]$, where $\sigma_{ess} = \left\{-\frac{1}{\alpha}\right\}$ and

  $$-\infty < \lambda_n^- < \lambda_n^+ < -1/\alpha \text{ with } \lambda_n^+ \to -1/\alpha, \lambda_n^- \to -\infty$$

  *(**Proof:** $z = \sqrt{1+\lambda\alpha}$ in char. eq. + comparison with $e^{2\frac{z^2-1}{\alpha z}} = -1$)*

This implies that:

- No finite subset of dominant eigenvalues $\Rightarrow$ PDE $\not\rightarrow$ ODE!!
- **Overdamping** phenomena