# Limit ODE
# of a nonlinear PDE model
# for a spring-mass-damper system

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**Celebrating the 80th birthday of Jack K. Hale**

# Outline

1. Model and objectives
2. Main result: limit ODE for the nonlinear PDE model
3. Sketch of the proof:

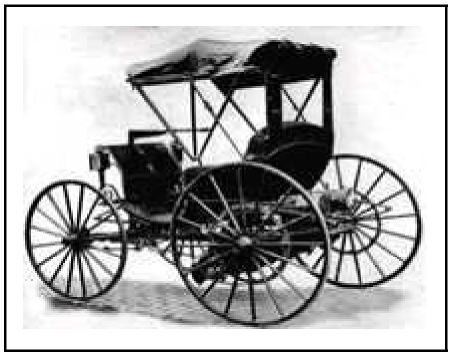
   exponentially attracting invariant manifolds
   (for a nonselfadjoint problem)

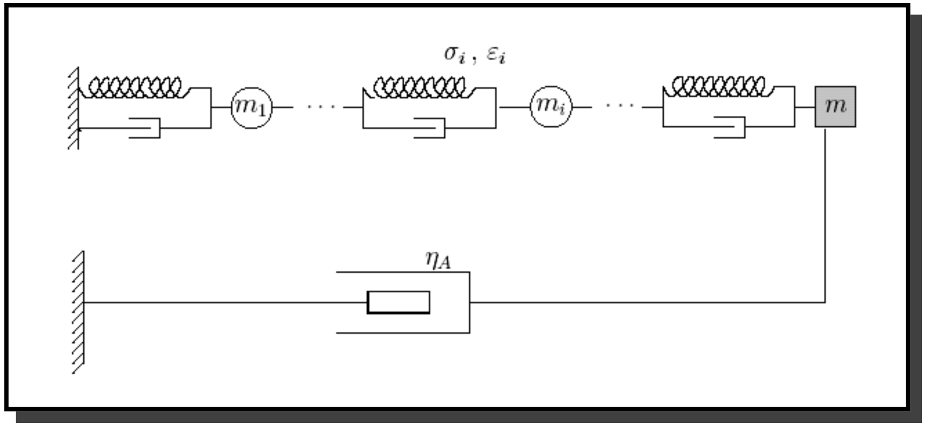
# 1. Model and objectives

- The system

**A damped**
**(and controlled)**
**spring-mass system**

Present in: automotion, seismic control, ...






Viscoelastic spring with a rigid moving mass and a viscous dashpot at the end $x = 1$.

$\Rightarrow$ **linear model** [a]

$+$

Imposing acceleration at $x = 0$ (*control*)

$\Rightarrow$ **nonlinear model** [b]

[a] M.P., J. Solà-Morales, JMAA (2004) and CPAA (2008)

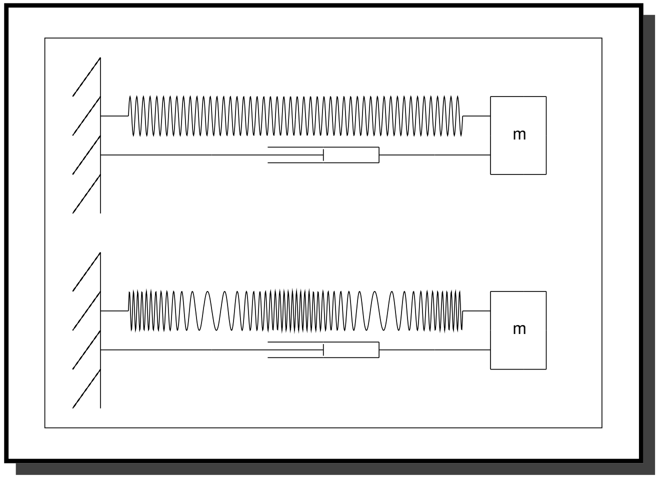
[b] M.P. *Large time dynamics of a nonlinear spring-mass-damper model*. Nonlinear Analysis (in press)

# PDE vs ODE

A classic model for a damped spring (**discrete** system):

$$m\,u''(t) + r\,u'(t) + k\,u(t) = 0$$

But ...

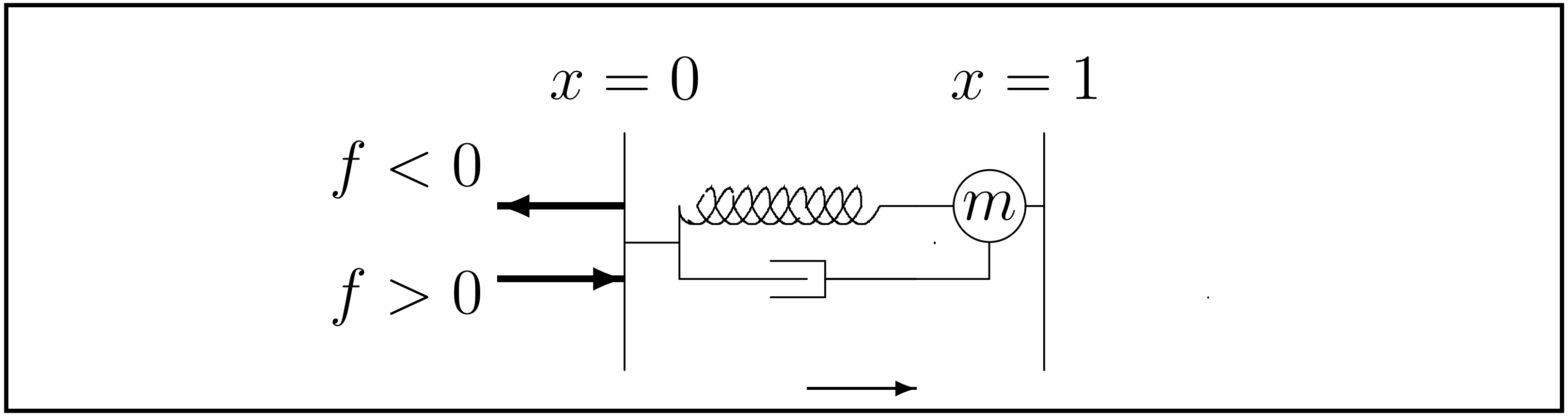



- differences in internal deformation?
- possible internal dissipation?

$$\Downarrow$$

PDE model (**continuous** system)

For the nonlinear case, we also impose an acceleration at $x = 0$:




$$u_{tt}(0) = \varepsilon f(\underbrace{u(1,t) - u(0,t)}_{displacement\ variation}, \underbrace{\frac{u_t(1,t) - u_t(0,t)}{\sqrt{\varepsilon}}}_{velocity\ variation})$$

where $f$ Lipschitz, bounded, *smooth enough*

# The model [a]

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) - \alpha\, u_{txx}(x,t) + \varepsilon f\left(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}\right) = 0 \\ u(0,t) = 0 \\ u_{tt}(1,t) = -\varepsilon\left[u_x + \alpha u_{tx} + r u_t\right](1,t) - \varepsilon f\left(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}\right) \end{cases}$$

$u(x,t) =$ displacement of the $x$-particle at time $t$.

Wave equation with **strong damping** and
**dynamical** boundary conditions,
with a **nonlocal nonlinearity**

[a] Linear model: M. Grobbelaar-van Dalsen, Appl. Analysis (1994).

## The model

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) - \alpha\, u_{txx}(x,t) + \varepsilon f\left(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}\right) = 0 \\ u(0,t) = 0 \\ u_{tt}(1,t) = -\varepsilon\left[u_x + \alpha u_{tx} + r u_t\right](1,t) - \varepsilon f\left(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}\right) \end{cases}$$

$\alpha > 0 \rightsquigarrow$ internal viscosity (spring)

$r > 0 \rightsquigarrow$ external damping (external dashpot)

$\varepsilon \geq 0 \rightsquigarrow$ inverse of the external mass

# 2. Main result

- **The objective:** compare ODE and PDE approaches

When $\varepsilon \to 0$ and $t \to \infty$,
there exists a limit ODE for the PDE model ?

- **Linear model**
  - Main tools: dominant eigenvalues
    + generalized conv. between operators [T.Kato].
  - Result (when $\varepsilon \to 0$).
    *If $\varepsilon$ is small enough, there exists a set of two dominant eigenvalues $\{\lambda_0^+(\varepsilon), \lambda_0^-(\varepsilon)\}$. Therefore, the PDE tends to a second order ODE:*

$$U''(t) + k_1 U'(t) + k_2 U(t) = 0.$$

(when $\alpha \to 0$, $\alpha >> 1$ different results)

# 2. Main result

- **The objective:** compare ODE and PDE approaches

When $\varepsilon \to 0$ and $t \to \infty$,
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- **Nonlinear model**
  - Main tools: invariant manifolds
    + generalized conv. between operators [T.Kato].
  - Result ...

**Theorem 1 (Existence of invariant manifolds).** If $\varepsilon$ small enough, the re-scaled system admits a **2D** exp. attract. inv. manifold

$$S_\varepsilon = \{V = (V_1, V_2) : V_1 = \eta_\varepsilon(V_2),\, V_2 \in H_2^\varepsilon\}$$

The flow on $S_\varepsilon$ is given by $V(t) = V_2(t) + \eta_\varepsilon(V_2(t))$, where $V_2(t)$:

$$\frac{d}{dt} V_2 = \left(\frac{1}{\sqrt{\varepsilon}}\ (A_\varepsilon|_{H_2^\varepsilon})\right) V_2 + P_2^\varepsilon \left[\frac{1}{\sqrt{\varepsilon}} F_\varepsilon\left(V_2 + \eta_\varepsilon(V_2)\right)\right], \quad V_2 \in H_2^\varepsilon.$$

Moreover, $\eta_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0$ in the $\mathcal{C}^1$ topology (and in an $\|\cdot\|_\varepsilon$ norm).

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Moreover, $\eta_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0$ in the $\mathcal{C}^1$ topology (and in an $\|\cdot\|_\varepsilon$ norm).

**Theorem 2 (The explicit limit ODE).** After a re-scaling in time, the nonlinear system converges in the $\mathcal{C}^1$ **topology** when $\varepsilon \to 0$ to:

$$\begin{cases} U'(t) = W(t) \\ W'(t) = -U(t) - f(U(t), W(t)) \end{cases}$$

If structurally stable $\Rightarrow$ topologically equivalent flows if $\varepsilon$ small enough.

Essentially, our main result says that ...

After a re-scaling in time, we have the nonlinear **limit ODE** ✓:

$$U'' + U + f(U, U') = 0\,, \quad U = U(t)$$

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⋆ ∗ ⋆

**Also as an inverse problem:**

$f$ as an external **control** at $x = 0$
to achieve a
**desired displacement**
(possible if solution of an ODE of the same type as the limit one).

# 3. Sketch of the proof

We write the PDE model as an evolution equation:

$$\begin{cases} \dfrac{dV}{dt} - A_\varepsilon V = F_\varepsilon(V), & t > 0 \\ \\ V(0) = V_0 \end{cases}$$

where

$(A_\varepsilon, \mathcal{D}(A_\varepsilon) \subset \mathcal{H}) \to$ linear operator

(analytic, not selfadjoint)

$F_\varepsilon : \mathcal{H} \longrightarrow \mathcal{H} \to$ nonlinearity

$\Rightarrow$ this is a well posed problem ✓

(in an appropriate functional framework)

# Exponentially attracting invariant manifolds

- **Definition.** Consider the system

$$\begin{cases} \dot{x} = Tx + h(x,y) \\ \dot{y} = By + g(x,y)\ ,\ (x,y) \in X \times Y \quad (eg.\ dim\, Y < \infty) \end{cases}$$

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- $S \subset X \times Y$ is an **invariant manifold** $\Leftrightarrow$ there exists $\eta : Y \to X$ s.t.

$$S = \{(x,y) \in X \times Y : x = \eta(y)\}$$

and exists $(x(\cdot), y(\cdot))$ solution s.t. $x(0) = x_0$, $y(0) = y_0$ and:

$$(x(t), y(t)) \in S\ \forall t \in \mathbb{R}, \text{ and } \forall\, (x_0, y_0) \in S$$

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$$(x(t), y(t)) \in S \ \forall t \in \mathbb{R}, \text{ and } \forall\, (x_0, y_0) \in S$$

- $S$ is **exponentially attracting** $\Leftrightarrow$ $\exists \gamma, K \geq 0$ s.t.:

$$\|x(t) - \eta(y(t))\| \leq K e^{-\gamma t} \|x(0) - \eta(y(0))\|$$

for all solution $(x(t), y(t))$.

## Theorem (existence of limit invariant manifolds) [*D.Henry'81; Carvalho (JDE'95); Carvalho, Lozada-Cruz (JMAA'06); Carbone'03*...]

Consider family of the weakly coupled systems:

$$\begin{cases} \dot{x} = T_\varepsilon\, x + h_\varepsilon(x,y), \text{ on } X_\varepsilon \ \ (T_\varepsilon \text{ sectorial}) \\ \dot{y} = B_\varepsilon\, y + g_\varepsilon(x,y), \text{ on } Y_\varepsilon \ \ (B_\varepsilon \text{ bounded}) \end{cases}$$

(H1-H4) $h_\varepsilon$ and $g_\varepsilon$ are Lipschitz and bounded (uniformly on $\varepsilon$);

(H5) $\| e^{T_\varepsilon t} w\|_{X_\varepsilon} \leq M_T\, e^{-\beta(\varepsilon)t} \|w\|_{X_\varepsilon}$, $t \geq 0$

(H6) $\| e^{B_\varepsilon t} z\|_{Y_\varepsilon} \leq M_B\, e^{\rho(\varepsilon)|t|} \|z\|_{Y_\varepsilon}$, $t \in \mathbb{R}$

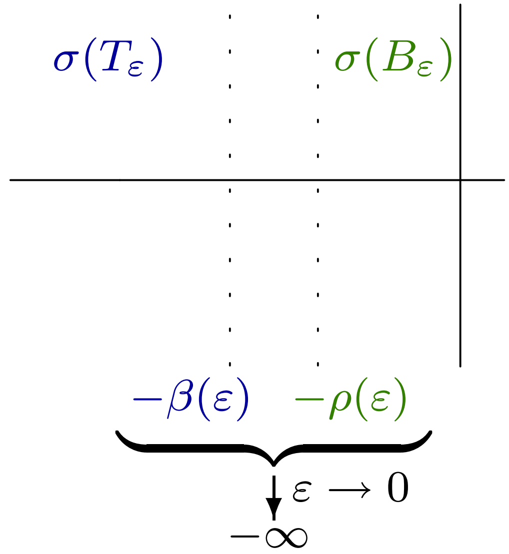
If $\beta(\varepsilon) - \rho(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \infty$, then for $\varepsilon << 1$:

$$\exists\, S_\varepsilon = \{(x,y) : x = \eta_\varepsilon(y),\ y \in Y_\varepsilon\} \text{ (exp. attract. inv. manifold)}$$

Moreover, if some regularity conditions hold, then

$$\eta_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} 0 \text{ in the } \mathcal{C}^1\text{-norm.}$$

# Main idea of the theorem:




- If spectrum is *sufficiently separated* in two parts ($\beta(\varepsilon) - \rho(\varepsilon) \to \infty$), everything goes to $-\infty$ except for a finite part, that gives the dynamics when $t \to \infty$:

$$\dot{y} = B_\varepsilon y + g_\varepsilon(\eta_\varepsilon(y), y) \quad \textbf{(ODE}\checkmark\textbf{)}$$

- Note: $\|\eta_\varepsilon\| \xrightarrow[\varepsilon \to 0]{\mathcal{C}^1} 0$. Allows to write the limit ODE **explicitly** ✓ .

# Sketch of the proof

- **STEP 1.** Preparation of the system:
  **re-scaling of time** $\Rightarrow$ appropriate separation of the spectrum.
- **STEP 2.** Verification of the **hypotheses** of previous theorem.
  (So, existence of $S_\varepsilon$ finite dimensional e.a.i.m. that tends to be flat in the $\mathcal{C}^1$ topology when $\varepsilon \to 0$. So, PDE tends to an ODE.)
- **STEP 3.** Obtention of the **explicit** limit ODE

Novelty (difficulty): $A_\varepsilon$ NOT selfadjoint

# STEP 1: re-scaling of time

$t \longrightarrow t\sqrt{\varepsilon}$ (acceleration of the system when $\varepsilon \longrightarrow 0$)

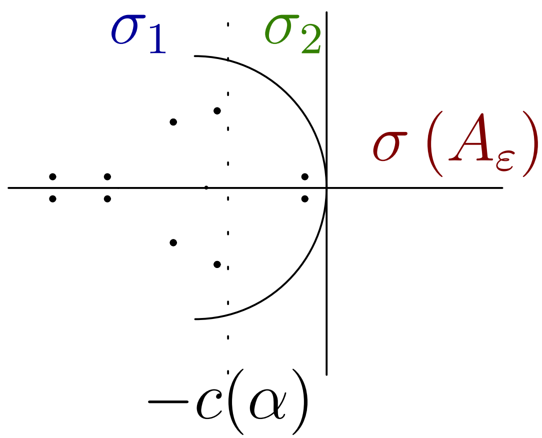
$$\frac{d}{dt}V - \frac{1}{\sqrt{\varepsilon}}A_\varepsilon V = \frac{1}{\sqrt{\varepsilon}}F_\varepsilon(V)$$

$$+$$

Spectral analysis of the linear operator $A_\varepsilon$

*(observe that* $\sigma\left(\frac{1}{\sqrt{\varepsilon}}A_\varepsilon\right) = \frac{1}{\sqrt{\varepsilon}}\sigma(A_\varepsilon)$

*but the eigenspaces are the same)*



# Separation of the spectrum ( $\Rightarrow$ of the space, projections, ... )

$$\sigma\left(\frac{1}{\sqrt{\varepsilon}}A_\varepsilon\right) = \sigma_1^\varepsilon \cup \sigma_2^\varepsilon$$

with

$$\sigma_1^\varepsilon = \sigma\left(\frac{1}{\sqrt{\varepsilon}}A_\varepsilon\right) \setminus \sigma_2^\varepsilon \Rightarrow Re\ (\sigma_1^\varepsilon) < \frac{-c(\alpha)}{\sqrt{\varepsilon}} \xrightarrow[\varepsilon\to 0]{} -\infty$$

$$\sigma_2^\varepsilon = \left\{\frac{\lambda_0^+(\varepsilon)}{\sqrt{\varepsilon}}, \frac{\lambda_0^-(\varepsilon)}{\sqrt{\varepsilon}}\right\} \xrightarrow[\varepsilon\to 0]{} \{\pm i\}$$

$$\Downarrow$$

$$\begin{cases} \dfrac{d}{dt}V_1 = \left(\frac{1}{\sqrt{\varepsilon}}\ (A_\varepsilon|_{H_1^\varepsilon})\right) V_1 + P_1^\varepsilon\left(\dfrac{1}{\sqrt{\varepsilon}}F_\varepsilon(V_1,V_2)\right) \ \ (\text{in } H_1^\varepsilon) \\ \\ \dfrac{d}{dt}V_2 = \left(\frac{1}{\sqrt{\varepsilon}}\ (A_\varepsilon|_{H_2^\varepsilon})\right) V_2 + P_2^\varepsilon\left(\dfrac{1}{\sqrt{\varepsilon}}F_\varepsilon(V_1,V_2)\right) \ \ (\text{in } H_2^\varepsilon) \end{cases}$$

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**2D !** ✓✓

## STEP 2: Verification of the hypotheses of the theorem

YES ✓

(**BUT**: big difficulty with H5 because $A_\varepsilon$ is NOT selfajoint
$\longrightarrow$ main difference with previous works)

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The hyphotesis 5: we want

$$\left\| e^{\frac{1}{\sqrt{\varepsilon}}\left(A_\varepsilon|_{H_1^\varepsilon}\right)t} V \right\|_\varepsilon \leq M e^{-\beta(\varepsilon)t} \|V\|_\varepsilon, \quad t > 0$$

with $M$ **independent** of $\varepsilon$. But $\frac{1}{\sqrt{\varepsilon}}\left(A_\varepsilon|_{H_1^\varepsilon}\right)$ is not selfadjoint.

What to do? Essentially, $M$ and $\beta(\varepsilon)$ depend on:

- the **sector** included in the resolvent set
- a certain **bound** of the resolvent operator in this sector

We can find a sector and bound independents of $\varepsilon$
for the resolvent of $\frac{1}{\sqrt{\varepsilon}}\left(A_\varepsilon|_{H_1^\varepsilon}\right)$ (if $\varepsilon$ small enough).

## How?

$A_\varepsilon|_{H_1^\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} A_0|_{H_1^0}$ in the generalized sense (*roughly speaking!*)

$\Rightarrow$ This allows us to will compare sectors and resolvents:
obtain a sector and bound for the resolvent **independents** of $\varepsilon$.

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**Therefore, we have**

- $M$ independent of $\varepsilon$
- $\beta(\varepsilon) = \frac{c(\alpha)}{\sqrt[4]{\varepsilon}}$ (and $\rho(\varepsilon) = 2\sqrt{2} + O(\sqrt{\varepsilon})$ ✓ )

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**So, theorem 1 is proved ...**

PDE tends to the equation on the exp. attr. inv. manifold $S_\varepsilon$ (2D✓)

$$\frac{d}{dt}V_2 = A_N\left(V_2\right) + P_2^\varepsilon\left[F_N\left(V_2 + \eta_\varepsilon(V_2)\right)\right], \quad V_2 \in H_2^\varepsilon$$

## STEP 3: Writing the explicit limit ODE

We want to calculate explicitly the limit of the ODE on $S_\varepsilon$.
(**BUT**: $A_\varepsilon$ not selfadjoint $\Rightarrow$ difficulty in projecting)

**How to proceed?**

As $A_\varepsilon \to A_0$ in generalized sense we have

$$\|P_2^\varepsilon - P_2^0\|_{\mathcal{H}} \xrightarrow[\varepsilon\to 0]{} 0$$

and

$$H_2^\varepsilon \cong H_2^0$$

(explicit isomorphism $Q_\varepsilon = P_0^{-1}$)

$\Rightarrow$ explicitly calculation of the limit equation.

So,

- Equation for $V_0 \in H_2^0$ (in a convenient form):

$$\frac{d}{dt} V_0 = \left( P_0 \frac{1}{\sqrt{\varepsilon}} A_\varepsilon Q_\varepsilon \right) V_0 + P_0 P_2^\varepsilon \left[ \frac{1}{\sqrt{\varepsilon}} F_\varepsilon \left( Q_\varepsilon V_0 + \eta_\varepsilon (Q_\varepsilon V_0) \right) \right]$$

- Add and subtract terms and take limits when $\varepsilon \to 0$

  (using convergence in norm of projections and $\eta_\varepsilon \to 0$ in $\mathcal{C}^1$):

$$\frac{d}{dt} V_0 = \left( P_0 \frac{1}{\sqrt{\varepsilon}} A_\varepsilon Q_\varepsilon \right) V_0 +$$

$$P_0 P_2^\varepsilon \left[ \frac{1}{\sqrt{\varepsilon}} F_\varepsilon \left( Q_\varepsilon V_0 + \eta_\varepsilon (Q_\varepsilon V_0) \right) \right] - P_2^0 P_2^0 \left[ \frac{1}{\sqrt{\varepsilon}} F_\varepsilon \left( Q_\varepsilon V_0 + \eta_\varepsilon (Q_\varepsilon V_0) \right) \right] +$$

$$P_2^0 P_2^0 \left[ \frac{1}{\sqrt{\varepsilon}} F_\varepsilon \left( Q_\varepsilon V_0 + \eta_\varepsilon (Q_\varepsilon V_0) \right) \right] - P_2^0 \frac{1}{\sqrt{\varepsilon}} F_\varepsilon (V_0) +$$

$$+ P_2^0 \frac{1}{\sqrt{\varepsilon}} F_\varepsilon (V_0)$$

The difference terms tend to 0 in $\mathcal{C}^1$-norm;
the rest, give the limit system in norm $\mathcal{C}^1$.

So ...

**limit ODE** (in the $\mathcal{C}^1$-norm):

$$\boxed{U'' + U + f(U, U') = 0}$$

(if structural stability $\Rightarrow$ PDE dynamics
when $\varepsilon \to 0$ and $t \to \infty$ time because $\mathcal{C}^1$ convergence)

**Therefore, theorem 2 is also proved**

**(and, hence, the main result) $\checkmark\checkmark$.**