Limit ODE and invariant manifolds in a nonlinear wave equation

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AIMS 2006
June 25-28, Poitiers

OUTLINE

1. Model and objectives

2. Main result: limit ODE for the nonlinear PDE model

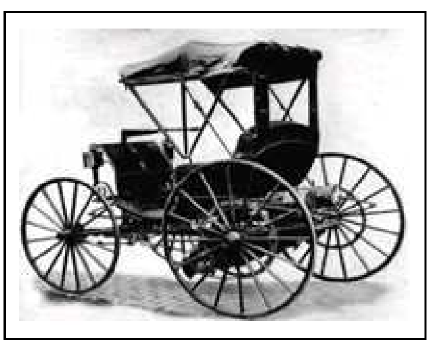
3. Proof and main tools: exponentially attracting invariant manifolds

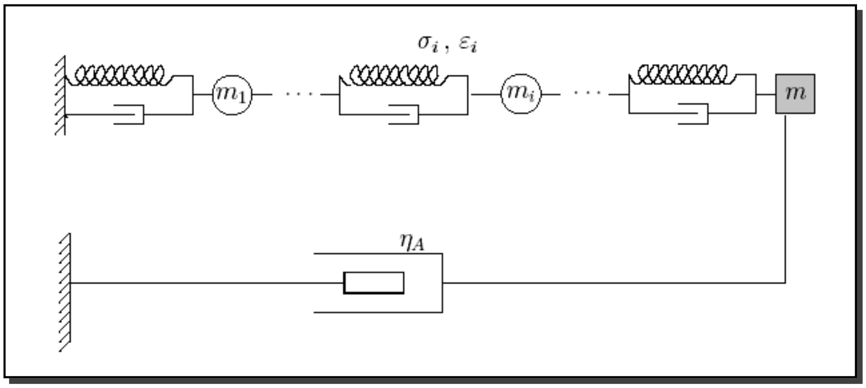
1. MODEL AND OBJECTIVES

- The system

A damped
(and controlled)
spring-mass system

Present in: automotion, seismic control, ...





Viscoelastic spring with a rigid moving mass
and a viscous dashpot at the end $x = 1$
$\Rightarrow$ **linear model**

$+$

Imposing acceleration at $x = 0$ (*control*)
$\Rightarrow$ **nonlinear model** *

*Preprint (2006) + In preparation

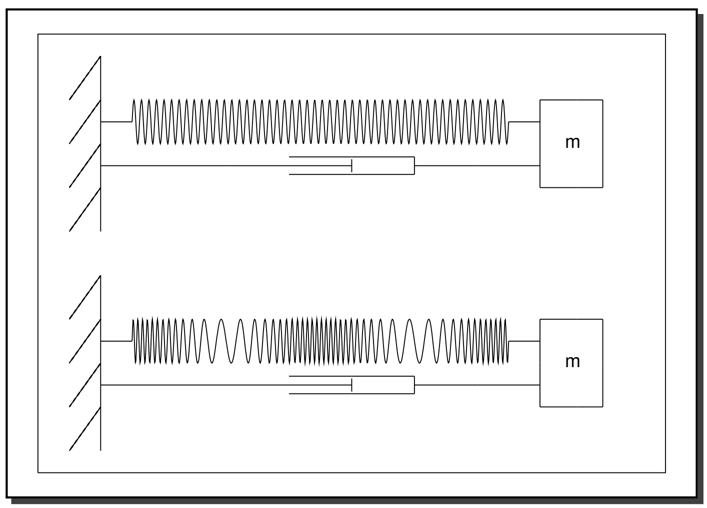
PDE vs ODE

A classic model for a damped spring:

$$m\,u''(t) + r\,u'(t) + k\,u(t) = 0$$

(spring as a **discrete** system)

But ...



- differences in internal deformation?
- possible internal dissipation?

$$\Downarrow$$

PDE model

The model *

$$
\begin{cases}
u_{tt}(x,t) - u_{xx}(x,t) - \alpha\, u_{txx}(x,t) + \varepsilon f\left(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}\right) = 0 \\
u(0,t) = 0 \\
u_{tt}(1,t) = -\varepsilon\left[u_x + \alpha u_{tx} + r u_t\right](1,t) - \varepsilon f\left(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}\right)
\end{cases}
$$

Wave equation with **strong damping** and **dynamical** boundary conditions, with a **nonlocal nonlinearity**

$u(x,t)$ = displacement of the x-particle at time t

$\alpha > 0 \rightsquigarrow$ internal viscosity (spring)

$r > 0 \rightsquigarrow$ external damping (external dashpot)

$\varepsilon \geq 0 \rightsquigarrow$ inverse of the external mass

*Linear model:

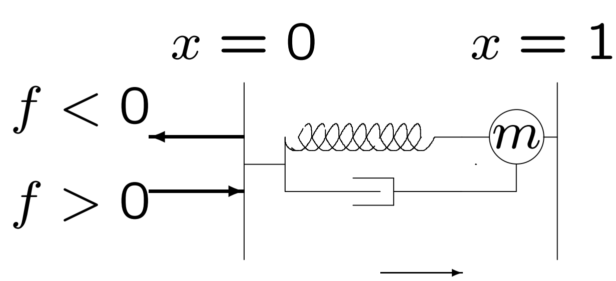
M. Grobbelaar-van Dalsen, Appl. Analysis, 53 (1994), 41-54.

M. P., J. Solà-Morales. JMAA (2004) and Preprint (2005)

We are considering an acceleration at $x = 0$ of the form:

$$u_{tt}(0) = \varepsilon f(\underbrace{u(1,t) - u(0,t)}_{displacement\ variation}, \underbrace{\frac{u_t(1,t) - u_t(0,t)}{\sqrt{\varepsilon}}}_{velocity\ variation})$$

where f Lipschitz, bounded, *smooth enough*



- The objective

Compare ODE and PDE approaches:
limit ODE when $t \to \infty$ for the PDE model?

$$\Downarrow$$

using **invariant manifolds**

$$\Downarrow$$

when $\varepsilon \to 0$ there exists a limit ODE ?

2. MAIN RESULT

Thm: After a re-scaling in time, the nonlinear system converges in the $\mathcal{C}^1$ topology when $\varepsilon \to 0$ to the nonlinear system:

$$\begin{cases} U' = W \\ \\ W' = -U - f(U,W) \end{cases}$$

$(U = U(t)$, $W = W(t))$.

If structurally stable $\Rightarrow$ topologically equivalent flows if ε sufficiently small.

So ...

after a re-scaling in time, we **DO** have the nonlinear **limit ODE** ✓:

$$U'' + U + f(U, U') = 0\,, \quad U = U(t)$$

⋆ ∗ ⋆

Also as an inverse problem:

f as an external **control** at $x = 0$ to achieve a **desired displacement** (possible if solution of an ODE of the same type as the limit one).

3. PROOF AND MAIN TOOLS

We write the PDE model as an evolution equation:

$$\begin{cases} \dfrac{dV}{dt} - A_\varepsilon V = F_\varepsilon(V), & t > 0 \\ \\ V(0) = V_0 \end{cases}$$

where

$(A_\varepsilon, \mathcal{D}(A_\varepsilon) \subset \mathcal{H}) \to$ linear operator
(not selfadjoint)

$F_\varepsilon : \mathcal{H} \longrightarrow \mathcal{H} \to$ nonlinearity

$\Rightarrow$ this is a well posed problem
(in an appropriate functional framework)

Definition

$T : \mathcal{D}(T) \subset X \to X$ sectorial, $Re\,\sigma(T) < 0$

$B : \mathcal{D}(B) \subset Y \to Y$ bounded

$h : X \times Y \to X$, $g : X \times Y \to Y$ cont. loc. Lips.

⋆ $S \subset X \times Y$ is an *invariant manifold* of

$$\begin{cases} \dot{x} = Tx + h(x,y) \\ \dot{y} = By + g(x,y) \end{cases} \quad (eg.\ dim\, Y < \infty)$$

if there exists $\sigma : Y \to X$ s.t.

$$S = \{(x,y) \in X \times Y : x = \sigma(y)\}$$

and $\forall\,(x_0, y_0) \in S$ exists $(x(\cdot), y(\cdot))$ solution s.t. $x(0) = x_0$, $y(0) = y_0$ and:

$$(x(t), y(t)) \in S \;\forall t \in \mathbb{R}$$

⋆ S is *exponentially attracting* if $\exists \gamma, K \geq 0$ s.t.:

$$\|x(t) - \sigma(y(t))\| \leq K e^{-\gamma t}\, \|x(0) - \sigma(y(0))\|$$

for all solution $(x(t), y(t))$.

Thm (existence of limit invariant manifolds)

D.Henry'81

Carvalho (JDE'95); Carvalho, Lozada-Cruz (JMAA'06); Carbone'03...

Consider the weakly coupled system:

$$\begin{cases} \dot{x} = T_\varepsilon \, x + h_\varepsilon(x, y) \\ \dot{y} = B_\varepsilon \, y + g_\varepsilon(x, y) \end{cases}$$

where:

(H1 -H4) h_ε and g_ε are Lipschitz and bounded (uniformly on ε);

(H5) $\| \, e^{T_\varepsilon \, t} w\|_{X_\varepsilon} \leq M_T \, e^{-\beta(\varepsilon)t} \|w\|_{X_\varepsilon}$, $t \geq 0$

(H6) $\| \, e^{B_\varepsilon \, t} z\|_{Y_\varepsilon} \leq M_B \, e^{\rho(\varepsilon)|t|} \|z\|_{Y_\varepsilon}$, $t \in \mathbb{R}$

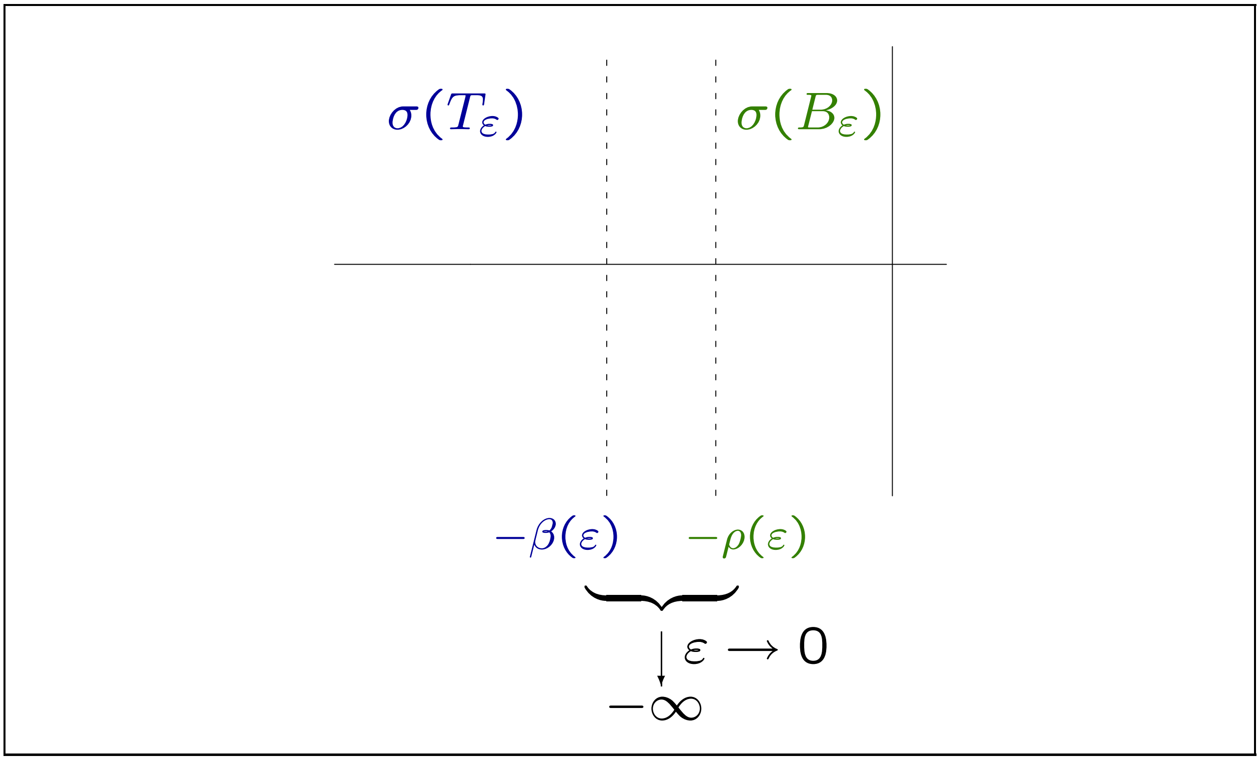
If $\beta(\varepsilon) - \rho(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \infty$ then if ε is small enough,

$$\exists S_\varepsilon = \{(x, y) : x = \sigma_\varepsilon(y), \, y \in Y_\varepsilon\} \text{ (e.a.i.m)}$$

Moreover, if some regularity conditions hold, then

$$\sigma_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} 0 \text{ in the } \mathcal{C}^1\text{-norm.}$$

Main idea of the theorem:



- If spectrum is *sufficiently separated* in two parts ($\beta(\varepsilon) - \rho(\varepsilon) \to \infty$), everything goes to $-\infty$ except for a finite part, that gives the dynamics when $t \to \infty$:

$$\dot{y} = B_\varepsilon y + g_\varepsilon(\sigma_\varepsilon(y), y) \quad (\textbf{ODE}\checkmark)$$

- Note: $\|\sigma_\varepsilon\| \xrightarrow[\varepsilon \to 0]{\mathcal{C}^1} 0$. Allows to write the limit ODE **explicitly** ✓ .

What are we going to do?

Re-scaling of time $\Rightarrow$ appropriate separation of the spectrum

$$\Downarrow$$

Re-scaled equation fulfills the hypothesis of Carvalho's theorem

$$\Downarrow$$

If ε small enough there exists S_ε e.a.i.m. that tends to be flat in the $\mathcal{C}^1$ topology when $\varepsilon \to 0$

$$\Downarrow$$

Explicit limit ODE

Novelty (difficulty): A_ε NOT selfadjoint

Preparing the problem

Re-scaling of time:

$$t \to t\sqrt{\varepsilon}$$

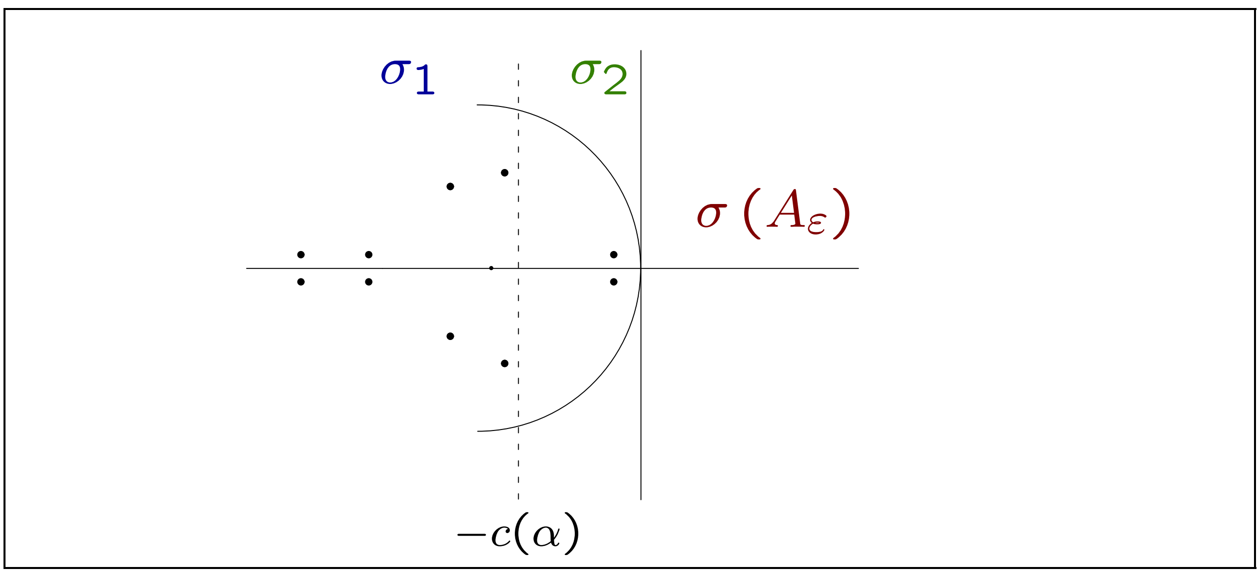
(acceleration of the system when $\varepsilon \to 0$)

$$\frac{d}{dt}V - \frac{1}{\sqrt{\varepsilon}}A_\varepsilon V = \frac{1}{\sqrt{\varepsilon}}F_\varepsilon(V)$$

$$+$$

Spectral analysis of the linear operator A_ε

(observe that $\sigma\left(\frac{1}{\sqrt{\varepsilon}}A_\varepsilon\right) = \frac{1}{\sqrt{\varepsilon}}\sigma(A_\varepsilon)$
but the eigenspaces are the same)



Separation of the spectrum
($\Rightarrow$ of the space, projections, ...)

$$\sigma\left(\frac{1}{\sqrt{\varepsilon}}A_\varepsilon\right) = \sigma_1^\varepsilon \cup \sigma_2^\varepsilon$$

with

$$\sigma_1^\varepsilon = \sigma\left(\frac{1}{\sqrt{\varepsilon}}A_\varepsilon\right) \setminus \sigma_2^\varepsilon \Rightarrow Re\ (\sigma_1^\varepsilon) < \frac{-c(\alpha)}{\sqrt{\varepsilon}} \underset{\varepsilon\to 0}{\longrightarrow} -\infty$$

$$\sigma_2^\varepsilon = \{\mu_0^+(\varepsilon), \mu_0^-(\varepsilon)\} \underset{\varepsilon\to 0}{\longrightarrow} \{\pm i\}$$

$$\Downarrow$$

$$\begin{cases} \dfrac{d}{dt}V_1 = \left(\frac{1}{\sqrt{\varepsilon}}\left(A_\varepsilon|_{H_1^\varepsilon}\right)\right)V_1 + P_1^\varepsilon\left(F_N(V)\right) \quad \text{(in } H_1^\varepsilon) \\ \dfrac{d}{dt}V_2 = \left(\frac{1}{\sqrt{\varepsilon}}\left(A_\varepsilon|_{H_2^\varepsilon}\right)\right)V_2 + P_2^\varepsilon\left(F_N(V)\right) \quad \text{(in } H_2^\varepsilon) \end{cases}$$

Observe that H_2^ε is a 2D-space!

- Verification of the hypothesis of Carvalho's thm

YES ✓

(**BUT**: big difficulty with H5 because A_ε is NOT selfajoint → main difference with Carvalho and collaborators work)

⇓

1. PDE tends to the equation on S_ε (2D✓)

$$\frac{d}{dt}V_2 = A_N(V_2) + P_2^\varepsilon\left[F_N(V_2 + \sigma_\varepsilon(V_2))\right], \; V_2 \in H_2^\varepsilon$$

2. explicit ODE when $\varepsilon \to 0$.

Writing the explicit limit ODE

We can now calculate the limit of the ODE in the invariant manifold S_ε.

BUT:

A_ε not selfadjoint $\Rightarrow$ difficulty in projecting

But:

$$H_2^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} H_2^0$$

- explicit isomorphism
- projections converging in norm
- projecting on H_2^0 is explicit

$$\Downarrow$$

equation on S_0 ($\cong S_\varepsilon$)

So ...

we write the equation on S_0
(change of variables)

$+$

limit of the coefficients in $\mathcal{C}^1$-norm using that $\sigma_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} 0$ and convergence in norm of projections

$\Downarrow$

limit ODE (in the $\mathcal{C}^1$-norm)

$$\boxed{U'' + U + f(U, U') = 0}$$

(if structural stability $\Rightarrow$ PDE dynamics when ε small and large time)

The hyphotesis 5

We want

$$\left\| e^{\frac{1}{\sqrt{\varepsilon}}\left(A_\varepsilon|_{H_1^\varepsilon}\right)t} V \right\| \le M e^{-\beta(\varepsilon)t} \|V\|, \quad t > 0$$

with M **independent** of ε. But $\frac{1}{\sqrt{\varepsilon}}\left(A_\varepsilon|_{H_1^\varepsilon}\right)$ is not selfadjoint

What to do? Essentially, M and $\beta(\varepsilon)$ depend on:

- the **sector** included in the resolvent set
- a certain **bound** of the resolvent operator in this sector

$$\Downarrow$$

Find a sector and bound independents of ε for the resolvent of $\frac{1}{\sqrt{\varepsilon}}\left(A_\varepsilon|_{H_1^\varepsilon}\right)$ (if ε small enough).

How?

Using that $A_\varepsilon|_{H_1^\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} A_0|_{H_1^0}$ in a *special sense*

$\Rightarrow$ We will compare sectors and resolvents.

We have

- M independent of ε
- $\beta(\varepsilon) = \frac{c(\alpha)}{\sqrt[4]{\varepsilon}}$

As $\rho(\varepsilon) = 2\sqrt{2} + O(\sqrt{\varepsilon})$, the hypothesis of the theorem are satisfied ✓ .