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Large time dynamics of a nonlinear spring–mass–damper model

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Abstract

In this paper we consider a nonlinear strongly damped wave equation as a model for a controlled spring–mass–damper system and give some results concerning its large time behaviour. It can be seen that the infinite dimensional system admits a two-dimensional attracting manifold where the equation is well represented by a classical nonlinear oscillations ODE, which can be exhibited explicitly. In contrast to other papers, this one applies Invariant Manifold Theory to a problem whose linear part is not self-adjoint.

MSC: 35B40; 35B42

Keywords: Nonlinear strongly damped wave equation; Exponentially attracting invariant manifolds; Limit ODE

1. Introduction

In the previous works [13,14], we have been interested in the large time behaviour of the solutions of the linear PDE problem

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) - \alpha u_{txx}(x,t) = 0 \\ u(0,t) = 0 \\ u_{tt}(1,t) = -\varepsilon \left[u_x(1,t) + \alpha u_{tx}(1,t) + r u_t(1,t)\right] \end{cases} \tag{1.1}$$

for $x \in (0,1)$, $t > 0$ and the parameters $\alpha, \varepsilon \geq 0$ and $r > 0$. In this model $u(x,t)$ represents the longitudinal displacement at time t of the x particle of a viscoelastic spring. This spring is attached at one end ($x = 0$) to a fixed wall and it is attached to a rigid moving body of mass $1/\varepsilon$ at the other end ($x = 1$). The possible spring inner viscosity or damping is represented by the parameter $\alpha \geq 0$. In this model we also consider an external dashpot acting on the spring movement through the mass at the end $x = 1$. This external damping is represented by $r > 0$. The details of the modelling of this system can be found in [13].

In these previous works [13,14] we considered whether or not the large time behaviour of the infinite dimensional dynamical system defined by (1.1) is well represented by a classical spring–mass–damper ODE with two degrees of freedom:

$$u''(t) + k_1\, u'(t) + k_0\, u(t) = 0. \tag{1.2}$$

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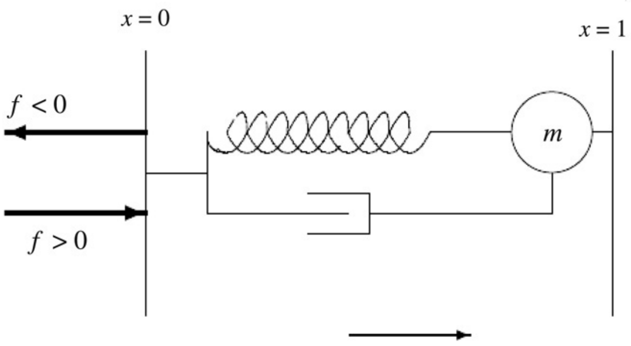

Fig. 1. The nonlinear spring–mass–damper system.

In this approach a careful analysis of the spectrum was carried out, especially analyzing the existence and behaviour of finite subsets of dominant eigenvalues. These sets are responsible for the large time dynamics of the solutions of the linear PDE problem. In [13] this analysis was done for when the parameter $\varepsilon \to 0$. We concluded the existence of two dominant eigenvalues when ε is small enough, which can be explicitly approximated. Hence, when $t \to \infty$ the solutions of (1.1) tend to solutions of a classical second-order ODE (1.2), whose coefficients can also be approximated explicitly in terms of the parameters of the system. In [14], the large time behaviour of (1.1) was studied with respect to changes in the spring inner viscosity $\alpha \geq 0$. More concretely, we studied three cases of physical and mathematical interest: $\alpha = 0$ (the purely elastic spring), α small (a spring with small inner viscosity) and α large (the overdamped case). In these three cases we obtained very different results, but in all of them the nonexistence of a limit ODE of the form (1.2) was proved.

In the present paper we study another problem also of interest, which consists of considering the same system as before but now with an imposed control acceleration at $x = 0$, the end previously fixed at a wall (see Fig. 1). The imposed acceleration is of the form

$$u_{tt}(0,t) = \varepsilon f\left(u(1,t) - u(0,t), \frac{u_t(1,t) - u_t(0,t)}{\sqrt{\varepsilon}}\right)$$

where f is a Lipschitz, bounded and sufficiently smooth function. That is, we consider that the acceleration that we impose depends on the position and the velocity differences between the two ends of the system: that means that depending on the present state of the spring, the acceleration will force the system to compress or expand in order to show a certain desired behaviour. The reason for the special scaling of ε will become clear in Section 5. Roughly speaking, the nonlinearity has to be appropriately re-scaled in order not to change the behaviour of the linear part.

In a reference system moving with the wall at $x = 0$ the model becomes

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) - \alpha u_{txx}(x,t) + \varepsilon f\left(u(1,t), \dfrac{u_t(1,t)}{\sqrt{\varepsilon}}\right) = 0 \\ u(0,t) = 0 \\ u_{tt}(1,t) = -\varepsilon\left[u_x + \alpha u_{tx} + r u_t\right](1,t) - \varepsilon f\left(u(1,t), \dfrac{u_t(1,t)}{\sqrt{\varepsilon}}\right) \end{cases} \tag{1.3}$$

with $x \in (0,1), t > 0$ and now with $\alpha, r > 0$, $\varepsilon \geq 0$. This is a nonlocal nonlinear strongly damped wave equation with dynamical boundary conditions. Observe that when $f \equiv 0$ we obtain the linear model (1.1).

Inspired by the results obtained for the linear problem (1.1) when ε is taken small, our goal is now to study whether Eq. (1.3) admits a nonlinear ordinary differential equation as a limit when $t \to \infty$ and $\varepsilon \to 0$, in a certain sense. We are going to see that in this case there exists such a limit equation that, with a re-scaled time, is the classical nonlinear oscillations equation:

$$U'' + U + f(U, U') = 0 \tag{1.4}$$

where $U = U(t)$. In fact, the reduction from (1.3) to (1.4) can be obtained by formal non-rigorous asymptotic methods: to capture the slow moving solutions, one can change to a new fast evolution variable $t \leftrightarrow t\sqrt{\varepsilon}$ and, taking in (1.3) only the leading order terms in ε, we obtain $u(x,t) = U(t)x$ with $U(t)$ satisfying (1.4). We emphasize that these are not true solutions, but only solutions to leading order. The goal of the present work is to obtain and explain this reduction from (1.3) to (1.4) by rigorous methods.

Actually, what happens is that the solutions of (1.3) are attracted, if ε is small enough, to a two-dimensional invariant manifold where the dynamics, after this re-scaling in time, corresponds to that of the Eq. (1.4). From the point of view of the controllability of the viscous spring–mass–damper system, f can be thought as an external control at $x = 0$ to achieve a desired dynamics of the system.

These attracting invariant manifolds (or inertial manifolds) are a very important tool in infinite dimensional dynamics (see Constantin, Foias, Nicolaenko and Temam in [6] for example). Our technical approach will use a result of Carvalho [4] that allows for an explicit limit equation for the dynamics inside the invariant manifold as $\varepsilon \to 0$. This can be done because the manifold tends to be flat.

The inertial manifolds have also been used in control problems by other authors like Rosa in [16]. In that paper this point of view is very well motivated, and it contains many references related to such problems. When comparing the results we announce here with the kinds of results of [16], it has to be said that our control is explicit and with a very natural physical meaning but, as a counterpart, it is only obtained as $\varepsilon \to 0$. Another difference is that our linear operators are not self-adjoint. More references on Invariant Manifold Theory will be given in Section 4.

This paper is organized as follows. In Section 2 we summarize the two main results which are the aim of this work. The first of them is the existence of a two-dimensional attracting manifold where the dynamics of (1.3) (appropriately re-scaled) takes place. The finite dimensional differential equation on this manifold is also obtained. This analysis is done in Section 4. In Section 5 we see the second result, that is the calculation and formal proof of the explicit limit ODE for this infinite dimensional problem. We also show in that section that the two dynamics (the PDE and ODE ones) are equivalent when $t \to \infty$ and $\varepsilon \to 0$. In order to do all this study, an abstract formulation of problem (1.3) is needed. This is provided in Section 3. Also in that section we introduce one of the main tools used in this work: the generalized convergence of operators. Finally, we have included an Appendix. In this Appendix we state some definitions and results related to semigroups, convergence between restrictions of operators and resolvent bounds for these restrictions. These results are used in Section 4.

2. Main results

In this section we state the main results of the present paper, which are Theorems 2.1 and 2.2. The details of the theory which is used as well as their proofs are developed in the sections below.

After the change in time $t \leftrightarrow t\sqrt{\varepsilon}$, we can write the problem (1.3) as the following evolution equation:

$$\begin{cases} \dfrac{\mathrm{d}V}{\mathrm{d}t} - \dfrac{1}{\sqrt{\varepsilon}} A_\varepsilon V = \dfrac{1}{\sqrt{\varepsilon}} F_\varepsilon(V), \quad t > 0 \\ V(0) = V_0 \end{cases} \tag{2.1}$$

where A_ε is the linear operator and F_ε gives the nonlinearity (see Section 3 for precise definitions). This is a well-posed initial value problem.

Using the results on the spectrum of A_ε, $\sigma(A_\varepsilon)$, given in [13], it is easy to see that the spectrum of the linear operator of the re-scaled Eq. (2.1) satisfies $\sigma\left(\frac{1}{\sqrt{\varepsilon}}A_\varepsilon\right) = \sigma_1^\varepsilon \cup \sigma_2^\varepsilon$ with

$$\operatorname{Re}(\sigma_1^\varepsilon) < -\frac{c(\alpha)}{\sqrt{\varepsilon}} \longrightarrow_{\varepsilon\to 0} -\infty, \qquad \sigma_2^\varepsilon = \left\{\mu_0^+(\varepsilon), \mu_0^-(\varepsilon)\right\} \longrightarrow_{\varepsilon \to 0} \pm \mathrm{i}$$

for a certain $c(\alpha) > 0$ and $\mu_0^\pm(\varepsilon) = \frac{\lambda_0^\pm(\varepsilon)}{\sqrt{\varepsilon}}$, where $\lambda_0^\pm(\varepsilon)$ are the perturbations of the double dominant eigenvalue of the operator A_0, $\lambda_0(0) = 0$. If we denote by H_1^ε and H_2^ε the eigenspaces corresponding to σ_1^ε and σ_2^ε, and by P_1^ε and P_2^ε the corresponding projections, Eq. (2.1) is equivalent to the system

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} V_1 = \left(\dfrac{1}{\sqrt{\varepsilon}}\left(A_\varepsilon|_{H_1^\varepsilon}\right)\right) V_1 + P_1^\varepsilon\left(\dfrac{1}{\sqrt{\varepsilon}} F_\varepsilon(V_1 + V_2)\right) \\ \dfrac{\mathrm{d}}{\mathrm{d}t} V_2 = \left(\dfrac{1}{\sqrt{\varepsilon}}\left(A_\varepsilon|_{H_2^\varepsilon}\right)\right) V_2 + P_2^\varepsilon\left(\dfrac{1}{\sqrt{\varepsilon}} F_\varepsilon(V_1 + V_2)\right) \end{cases} \tag{2.2}$$

with $V_1 \in H_1^\varepsilon$, $V_2 \in H_2^\varepsilon$. As we said before, our point of view is based on the results of Carvalho [4], also appearing in other works (see references in Section 4). Nevertheless, the main difficulty in our problem relies on the fact that these results have been applied to a non-self-adjoint case (as we will see in Sections 4 and 5) and, to our knowledge, for the first time. The first main result of the present paper is the following:

Theorem 2.1 (*Existence of Invariant Manifolds*). *For ε sufficiently small, system (2.2) admits an exponentially attracting invariant manifold of dimension 2 given by the graph of a function η_ε:*

$$S_\varepsilon = \{V = (V_1, V_2) : V_1 = \eta_\varepsilon(V_2),\ V_2 \in H_2^\varepsilon\}.$$

The flow on S_ε is given by $V(t) = V_2(t) + \eta_\varepsilon(V_2(t))$, where $V_2(t)$ is a solution of

$$\frac{\mathrm{d}}{\mathrm{d}t} V_2 = \left(\frac{1}{\sqrt{\varepsilon}}\left(A_\varepsilon|_{H_2^\varepsilon}\right)\right) V_2 + P_2^\varepsilon\left[\frac{1}{\sqrt{\varepsilon}} F_\varepsilon\left(V_2 + \eta_\varepsilon(V_2)\right)\right],\quad V_2 \in H_2^\varepsilon. \tag{2.3}$$

Moreover, $\eta_\varepsilon : H_2^\varepsilon \to H_1^\varepsilon$ tends to zero in the topology of $\mathcal{C}^1(H_2^\varepsilon, H_1^\varepsilon; \|\cdot\|_\varepsilon)$ when $\varepsilon \to 0$.

(The norm $\|\cdot\|_\varepsilon$ depends on ε and will be defined in the following section.) This result is seen in Section 4. This allows us to say that the solutions of problem (1.3) conveniently re-scaled are attracted to this two-dimensional manifold S_ε, where the dynamics corresponds to that of Eq. (2.3). Observe also that this attracting manifold tends to be flat when $\varepsilon \to 0$, because $\eta_\varepsilon \to 0$ in the $\mathcal{C}^1$ topology. This is a very important result in the theorem of existence of Carvalho [4] that is used here, because this asymptotic behaviour allows us to compute explicitly the limit of the ODE in the S_ε when $\varepsilon \to 0$. This is done with the next theorem, whose derivation is the purpose of Section 5, and which is the second main result of this paper.

Theorem 2.2 (*The Explicit Limit ODE*). *Eq. (2.3) converges in the $\mathcal{C}^1$ topology when $\varepsilon \to 0$ to the system*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u(1) \\ w(1) \end{pmatrix} = \begin{pmatrix} w(1) \\ -u(1) - f(u(1), w(1)) \end{pmatrix}. \tag{2.4}$$

In particular, if (2.4) is structurally stable, then, for a sufficiently small ε, the flow on the manifold given by (2.3) is topologically equivalent to that of (2.4).

Observe that the system (2.4) can be written as the nonlinear second-order differential equation

$$U'' + U + f(U, U') = 0. \tag{2.5}$$

We have seen that, after re-scaling the time, solutions of (1.3) tend to solutions of (2.5) when $t \to \infty$ and $\varepsilon \to 0$. As we said, this can also be interpreted in terms of a control problem. The conclusion in this case is that when the desired behaviour is the solution of an equation of the form (2.5) then it can be obtained from (1.3) when we apply to our system an acceleration given by f.

It is important to remark that the convergence between the nonlinear PDE and the nonlinear ODE is in the $\mathcal{C}^1$ topology and not only in the $\mathcal{C}^0$ sense. This allows us to say that, if the system is structurally stable, the dynamics of the PDE and the dynamics of the ODE are equivalent (for large times and small ε). This cannot be assured if the system is not structurally stable or the convergence is just in the $\mathcal{C}^0$ topology.

Further remarks will be given in the sections below.

3. Functional framework and tools

In this section we describe the appropriate functional framework where problem (1.3) is well-posed. We also describe one of the main analytical tools that will be used later, the generalized convergence between operators. Part of the contents of this section can be found in the previous works [13,14], but we have included it here for a better comprehension of the present work.

Let us define the following subspaces:

$$X_0 = L^2(0, 1) \times \mathbb{C}$$
$$X_1 = \{(u, \gamma) \in H^1(0, 1) \times \mathbb{C}, u(0) = 0,\ u(1) = \gamma\} \subset H^1(0, 1) \times \mathbb{C}.$$

We define the total space $\mathcal{H} = X_1 \times X_0$, which is a Hilbert space with the norm

$$\left\| \begin{pmatrix} (u, u(1)) \\ (v, \beta) \end{pmatrix} \right\|_{\mathcal{H}}^2 = \int_0^1 |u_x|^2 \, \mathrm{d}x + \int_0^1 |v|^2 \, \mathrm{d}x + |\beta|^2 \tag{3.1}$$

which is equivalent to the natural norm defined in $\mathcal{H}$. But for $\varepsilon > 0$ we can also define a family of norms in $\mathcal{H}$, each of them being denoted by $\|\cdot\|_\varepsilon$:

$$\left\| \begin{pmatrix} (u, u(1)) \\ (v, \beta) \end{pmatrix} \right\|_{\varepsilon}^2 = \int_0^1 |u_x|^2 \, \mathrm{d}x + \int_0^1 |v|^2 \, \mathrm{d}x + \frac{1}{\varepsilon}|\beta|^2. \tag{3.2}$$

Notice that for any $\varepsilon > 0$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_\varepsilon$ are equivalent norms in $\mathcal{H}$. Also, $\|\cdot\|_\varepsilon^2$ has physical meaning as the total energy of the spring.

Now, Eq. (1.3) can be written as the following evolution equation:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} V - A_\varepsilon V = F_\varepsilon(V), t > 0 \\ V(0) = V_0 \end{cases} \tag{3.3}$$

where A_ε is the linear operator defined as

$$A_\varepsilon \begin{pmatrix} (u, u(1)) \\ (v, v(1)) \end{pmatrix} = \begin{pmatrix} (v, v(1)) \\ ((u + \alpha v)_{xx}, -\varepsilon(u + \alpha v)_x(1) - \varepsilon r v(1)) \end{pmatrix}$$

with domain

$$\mathcal{D}(A_\varepsilon) = \left\{ \begin{pmatrix} (u, u(1)) \\ (v, v(1)) \end{pmatrix} \in X_1 \times X_1, \ (u + \alpha v) \in H^2(0, 1) \right\} \subset \mathcal{H}$$

and the nonlinearity $F_\varepsilon : \mathcal{H} \longrightarrow \mathcal{H}$ is defined as

$$F_\varepsilon \begin{pmatrix} (u, u(1)) \\ (v, \beta) \end{pmatrix} = \begin{pmatrix} (0, 0) \\ \left(-\varepsilon f\left(u(1), \dfrac{\beta}{\sqrt{\varepsilon}}\right), -\varepsilon f\left(u(1), \dfrac{\beta}{\sqrt{\varepsilon}}\right)\right) \end{pmatrix}.$$

We denote by A_0 the operator A_ε for the limit case $\varepsilon = 0$.

It should be said that the fact of having the possibility of choosing the most appropriate norm between $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_\varepsilon$ at each moment will be very important in this work. This is true even though $\|\cdot\|_\varepsilon$ may seem to be problematic when $\varepsilon \to 0$, which is the limit that we are interested in.

Theorem 3.1. *The following results are true:*

(a) A_ε *is a dissipative operator in the* $\|\cdot\|_\varepsilon$ *norm when* $\varepsilon > 0$ *and* $\alpha, r > 0$. *And it generates an analytic semigroup when* $\varepsilon \geq 0$ *and* $\alpha, r > 0$.

(b) *The nonlinear Eq.* (3.3) *for* $\varepsilon, \alpha > 0$ *admits a unique solution in* $\mathcal{H}$ *for each initial condition* $V_0 \in \mathcal{H}$.

Proof. Part (a) can be seen from [13]. For part (b) it is only necessary to prove that F_ε is Lipschitz in $\mathcal{H}$ (see [11]). This is true because f is a Lipschitz function. □

As we have said in the introduction we are interested in the large time behaviour of solutions of the equation that is now in the form (3.3) when $\varepsilon \to 0$. In order to find this, a careful study of the operator spectrum will be made. This is why we introduce here the notion of generalized convergence between operators, that can be found with more details in [9]. This convergence is a generalization of the convergence in norm but for operators that may be unbounded and turns out to be the appropriate one for comparing the spectra of operators. Essentially, if two operators are close enough in the generalized sense, the distance between corresponding compact subsets of the spectra will also be small (see Theorem 3.16 of chapter IV of Kato [9] for the details). In our case, we have the following result.

Theorem 3.2. *The operators* A_ε *converge in the generalized sense to the operator* A_0 *when* $\varepsilon \to 0$ *in the* $\|\cdot\|_{\mathcal{H}}$ *norm.*

The proof of this theorem can be found in [13].

4. The existence of exponentially attracting invariant manifolds

The aim of this section is to prove Theorem 2.1, that is the existence of an exponentially attracting invariant manifold of dimension 2 for the Eq. (2.1) if ε is sufficiently small. The existence of such an attracting invariant manifold will allow us to describe explicitly, in the next section, the large time dynamics of our problem.

The theory of exponentially attracting invariant manifolds (or inertial manifolds) used in this section is based on the study of Henry in [10]. These manifolds allow us to describe the main elements of the dynamics of the system that we are considering through a limit ordinary differential equation. This theory has been the origin of several works during the last few years. To mention just some of them, see the works of Mora and Solà-Morales in [15] on invariant manifolds and, as we mentioned in Section 1, also the work of Rosa in [16] on inertial manifolds used in control problems.

We will use the point of view of the works Carvalho [4], Arrieta and Carvalho [1], Carbone [2] and, more recently, Carbone and Ruas-Filho [3] or Carvalho and Lozada-Cruz [5], for instance. In these works, a theorem of existence of an exponentially attracting invariant manifold is given. But it should be said that the main new contribution of their approach is the fact that the attracting manifold tends to be flat. This allows one to obtain an explicit limit differential equation, which is also done in all of these works. This theorem will also be used in our case. As we said before, one of the contributions of the present work is that this theory is applied to a non-self-adjoint case and, to our knowledge, for the first time.

For a better comprehension of the present paper, the main elements of this theory are included below. Let X and Y be Banach spaces and $T : \mathcal{D}(T) \subset X \to X$ be a sectorial operator with $Re\,\sigma(T) < 0$. Let $B : \mathcal{D}(B) \subset Y \to Y$ be the generator of a $\mathcal{C}^0$-group of bounded linear operators.

Definition 4.1. Let us consider $h : X \times Y \to X$ and $g : X \times Y \to Y$ continuous and locally Lipschitz functions. A set $S \subset X \times Y$ is called an *invariant manifold* for the differential equation

$$\begin{cases} \dot{x} = Tx + h(x, y) \\ \dot{y} = By + g(x, y) \end{cases} \tag{4.1}$$

if there exists $\eta : Y \to X$ such that $S = \{(x, y) \in X \times Y : x = \eta(y)\}$ and for all $(x_0, y_0) \in S$ there exists $(x(\cdot), y(\cdot))$, a solution of (4.1) for all $t \in \mathbb{R}$, with $x(0) = x_0$ and $y(0) = y_0$, such that $(x(t), y(t)) \in S$ for all $t \in \mathbb{R}$.

The invariant manifold S is also called *exponentially attracting* if there exist positive constants γ and K such that

$$\|x(t) - \eta(y(t))\|_X \le K e^{-\gamma t} \|x(0) - \eta(y(0))\|_X$$

for all $(x(t), y(t))$ being a solution of (4.1).

The next theorem provides sufficient conditions for the existence of an exponentially attracting invariant manifold for a family of equations of the type of (4.1).

Theorem 4.2 (*[4,1,2,5]*). *Let X_ε and Y_ε be a family of Banach spaces, $T_\varepsilon : \mathcal{D}(T_\varepsilon) \subset X_\varepsilon \to X_\varepsilon$ be a family of sectorial operators and $B_\varepsilon : \mathcal{D}(B_\varepsilon) \subset Y_\varepsilon \to Y_\varepsilon$ be a family of generators of $\mathcal{C}^0$-groups of bounded linear operators. Suppose that the following hypotheses are satisfied:*

(H1) *$h_\varepsilon : X_\varepsilon \times Y_\varepsilon \to X_\varepsilon$ and $g_\varepsilon : X_\varepsilon \times Y_\varepsilon \to Y_\varepsilon$ are continuous and locally Lipschitz functions such that*

$$\|h_\varepsilon(x, y) - h_\varepsilon(x', y')\|_{X_\varepsilon} \le L_h \left(\|x - x'\|_{X_\varepsilon} + \|y - y'\|_{Y_\varepsilon}\right)$$
$$\|h_\varepsilon(x, y)\|_{X_\varepsilon} \le N_h$$
$$\|g_\varepsilon(x, y) - g_\varepsilon(x', y')\|_{Y_\varepsilon} \le L_g \left(\|x - x'\|_{X_\varepsilon} + \|y - y'\|_{Y_\varepsilon}\right)$$
$$\|g_\varepsilon(x, y)\|_{Y_\varepsilon} \le N_g$$

for all $(x, y), (x', y') \in X_\varepsilon \times Y_\varepsilon$ and some L_h, N_h, L_g and $N_g > 0$.

(H2) *The semigroup generated by T_ε satisfies that*

$$\|e^{T_\varepsilon t} w\|_{X_\varepsilon} \le M_T\, e^{-\beta(\varepsilon) t} \|w\|_{X_\varepsilon}, \quad t \ge 0$$

for any $w \in X_\varepsilon$ and some positive M_T and $\beta(\varepsilon)$.

(H3) *The operator* B_ε *fulfills that*

$$\|e^{B_\varepsilon t} z\|_{Y_\varepsilon} \le M_B\, e^{\rho(\varepsilon)|t|} \|z\|_{Y_\varepsilon}, \quad t \in \mathbb{R}$$

for any $z \in Y_\varepsilon$ *and some positive* M_B *and* $\rho(\varepsilon)$.

We now consider the following weakly coupled system:

$$\begin{cases} \dot{x} = T_\varepsilon\, x + h_\varepsilon(x, y) \\ \dot{y} = B_\varepsilon\, y + g_\varepsilon(x, y). \end{cases} \tag{4.2}$$

If $\beta(\varepsilon) - \rho(\varepsilon) \to \infty$ *when* $\varepsilon \to 0$, *then for a sufficiently small* ε *there exists an exponentially attracting invariant manifold for Eq.* (4.2)*:*

$$S_\varepsilon = \{(x, y) : x = \eta_\varepsilon(y),\ y \in Y_\varepsilon\}.$$

This $\eta_\varepsilon : Y_\varepsilon \to X_\varepsilon$ *satisfies the following properties:*

(a) *if we call* $s(\varepsilon) = \sup_{y \in Y_\varepsilon} \|\eta_\varepsilon(y)\|_{X_\varepsilon}$, *then* $s(\varepsilon) \to 0$ *when* $\varepsilon \to 0$,
(b) $\|\eta_\varepsilon(y) - \eta_\varepsilon(z)\|_{X_\varepsilon} \le l(\varepsilon)\|y - z\|_{Y_\varepsilon}$ *and* $l(\varepsilon) \to 0$ *when* $\varepsilon \to 0$.

Moreover, η_ε *is smooth if* h_ε *and* g_ε *are sufficiently smooth. And the derivative of* η_ε, $D\eta_\varepsilon$, *satisfies that*

$$\sup_{y \in Y_\varepsilon} \|D\eta_\varepsilon(y)\|_{L(Y_\varepsilon, X_\varepsilon)} \le l(\varepsilon).$$

Remark 4.3. We have reproduced the theorem in the form that will be needed in the sequel. But the theorem can also be found in the above references for a more general situation, where the nonlinearities are defined only in domains involving fractional powers of the operators.

If we have an exponentially attracting invariant manifold with the properties given by Theorem 4.2 we can consider the equation

$$\dot{y} = B_\varepsilon y + g_\varepsilon(\eta_\varepsilon(y), y), \quad y \in Y_\varepsilon \tag{4.3}$$

as the limit for the system (4.2) when $t \to \infty$ if ε is small enough. In this sense, the existence of such a manifold can be thought of as the analogue of the existence of finite subsets of dominant eigenvalues in the linear case (see [13]).

The main idea of this result of Theorem 4.2 consists of separating the whole space into two parts such that in one of these the spectrum tends to $-\infty$. Consequently, the dynamics of the whole equation will be given by the rest of the spectrum which, in our case, results in an eigenspace of dimension 2. That means that we will obtain a (nonlinear) second-order ODE as the limit for the nonlinear PDE. And as S_ε tends to be flat, we will be able to compute explicitly the limit equation.

We proceed now to write Eq. (3.3) in an appropriate way such that the hypotheses of Theorem 4.2 are fulfilled. For that, it is convenient to re-scale the time to capture the slow moving solutions. So we change to the new fast evolution variable $t \leftrightarrow t\sqrt{\varepsilon}$. Now, Eq. (3.3) becomes

$$\begin{cases} \dfrac{dV}{dt} - \dfrac{1}{\sqrt{\varepsilon}} A_\varepsilon V = \dfrac{1}{\sqrt{\varepsilon}} F_\varepsilon(V), \quad t > 0 \\ V(0) = V_0. \end{cases} \tag{4.4}$$

Notice that there is a correspondence between the sets of eigenvalues: λ is an eigenvalue of A_ε if and only if $\mu = \lambda/\sqrt{\varepsilon}$ is an eigenvalue of $(1/\sqrt{\varepsilon})\, A_\varepsilon$. But observe that the corresponding eigenspaces are the same.

We want to decompose (4.4) as a system of the type of Theorem 4.2 for small ε. So, as is done in [10], we consider the spectral sets

$$\sigma_2^\varepsilon = \{\mu_0^+(\varepsilon), \mu_0^-(\varepsilon)\} = \left\{ \frac{\lambda_0^+(\varepsilon)}{\sqrt{\varepsilon}}, \frac{\lambda_0^-(\varepsilon)}{\sqrt{\varepsilon}} \right\} \subset \sigma\left(\frac{1}{\sqrt{\varepsilon}} A_\varepsilon\right),$$

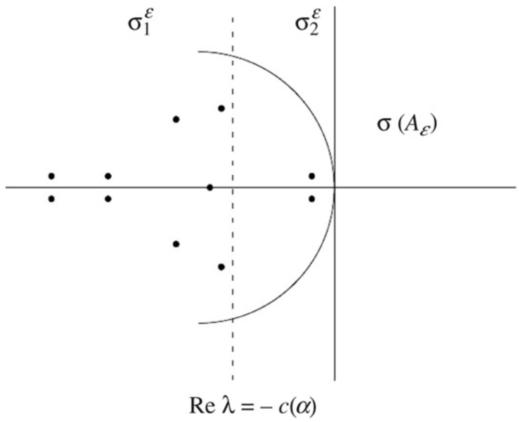

Fig. 2. Separation of the spectrum of A_ε when ε is small.

which is bounded, and

$$\sigma_1^\varepsilon = \sigma\left(\frac{1}{\sqrt{\varepsilon}}\,A_\varepsilon\right) \setminus \sigma_2^\varepsilon \subset \sigma\left(\frac{1}{\sqrt{\varepsilon}}\,A_\varepsilon\right),$$

where $\lambda_0^\pm(\varepsilon)$ are the dominant eigenvalues of A_ε when ε is small enough (see [13] and Fig. 2).

We call P_1^ε and P_2^ε the projections associated with each spectral set and

$$H_1^\varepsilon = P_1^\varepsilon(\mathcal{H}), \qquad H_2^\varepsilon = P_2^\varepsilon(\mathcal{H}). \tag{4.5}$$

Observe that, in particular, H_2^ε is a two-dimensional subspace generated by Ψ_ε^1 and Ψ_ε^2, the eigenfunctions associated with σ_2^ε. These eigenfunctions can be computed explicitly:

$$\Psi_\varepsilon^1 = \begin{pmatrix} (\phi_\varepsilon^1(x), \phi_\varepsilon^1(1)) \\ (\lambda_0^+(\varepsilon)\phi_\varepsilon^1(x), \lambda_0^+(\varepsilon)\phi_\varepsilon^1(1)) \end{pmatrix}, \qquad \Psi_\varepsilon^2 = \begin{pmatrix} (\phi_\varepsilon^2(x), \phi_\varepsilon^2(1)) \\ (\lambda_0^-(\varepsilon)\phi_\varepsilon^2(x), \lambda_0^-(\varepsilon)\phi_\varepsilon^2(1)) \end{pmatrix} \tag{4.6}$$

where

$$\phi_\varepsilon^1(x) = \sin C(\varepsilon)x, \phi_\varepsilon^2(x) = \sin \overline{C(\varepsilon)}x \quad \text{and} \quad C(\varepsilon) = \frac{\lambda_0^+(\varepsilon)\mathrm{i}}{\sqrt{1+\lambda_0^+(\varepsilon)\alpha}}. \tag{4.7}$$

Remark 4.4. For the limit case $\varepsilon = 0$, the subspace H_2^0 corresponding to the double dominant eigenvalue of A_0, $\lambda_0^+(0) = \lambda_0^-(0) = 0$, is spanned by the generalized eigenfunctions

$$\left\{ \begin{pmatrix} (x,1) \\ (0,0) \end{pmatrix}, \begin{pmatrix} (0,0) \\ (x,1) \end{pmatrix} \right\} \tag{4.8}$$

which do not correspond to eigenfunctions of $(1/\sqrt{\varepsilon})A_\varepsilon$, an operator which is not defined when $\varepsilon = 0$.

So this decomposition of $\sigma\left((1/\sqrt{\varepsilon})\,A_\varepsilon\right)$ induces a decomposition of the whole space into two invariant subspaces: $\mathcal{H} = H_1^\varepsilon \oplus H_2^\varepsilon$, with $\dim H_2^\varepsilon = 2 < \infty$. A decomposition of the whole operator is also obtained: $\left(1/\sqrt{\varepsilon}\right) A_\varepsilon|_{H_2^\varepsilon}$ is bounded and with spectrum σ_2^ε and, as A_ε is sectorial, so is $\left(1/\sqrt{\varepsilon}\right) A_\varepsilon|_{H_1^\varepsilon}$, whose spectrum is σ_1^ε. Moreover, the results on the spectrum of A_ε obtained in [13] imply that

$$\sigma_2^\varepsilon = \{\mu_0^\pm(\varepsilon)\} = \left\{\frac{\lambda_0^\pm(\varepsilon)}{\sqrt{\varepsilon}}\right\} \longrightarrow_{\varepsilon\to 0} \{\pm i\} \quad \text{and} \quad \mathrm{Re}(\sigma_1^\varepsilon) < \frac{-c(\alpha)}{\sqrt{\varepsilon}} \longrightarrow_{\varepsilon\to 0} -\infty$$

where $0 < c(\alpha) < \min\{1/\alpha,\ \alpha\pi^2/2\}$. Some comments have to be made.

Remark 4.5. The fact that $A_\varepsilon \to A_0$ when $\varepsilon \to 0$ in the generalized sense (as is said in Theorem 3.2) implies the convergence in norm of the corresponding projections, that is,

$$\lim_{\varepsilon\to 0} \|P_i^\varepsilon - P_i^0\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} = 0, \quad i = 1, 2.$$

We can compute the projection P_2^0 explicitly:

$$P_2^0 : \mathcal{D}(A_\varepsilon) \subset \mathcal{H} = H_1^0 \oplus H_2^0 \longrightarrow H_2^0$$
$$\begin{pmatrix}(u, u(1))\\(v, v(1))\end{pmatrix} \longrightarrow \begin{pmatrix}(u(1)x, u(1))\\(v(1)x, v(1))\end{pmatrix}.$$

Also, the generalized convergence between A_ε and A_0 implies that H_i^ε and H_i^0, $i = 1, 2$, are isomorphic subspaces if ε is small enough. Actually, the isomorphism between H_2^ε and H_2^0 turns out to be explicit, as we will see below.

After all these considerations, we can project Eq. (4.4) onto each invariant subspace and hence (4.4) can be written as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} V_1 = \left(\dfrac{1}{\sqrt{\varepsilon}}\left(A_\varepsilon|_{H_1^\varepsilon}\right)\right) V_1 + h_\varepsilon(V_1, V_2) \\ \dfrac{\mathrm{d}}{\mathrm{d}t} V_2 = \left(\dfrac{1}{\sqrt{\varepsilon}}\left(A_\varepsilon|_{H_2^\varepsilon}\right)\right) V_2 + g_\varepsilon(V_1, V_2) \end{cases} \tag{4.9}$$

where

$$h_\varepsilon(V_1, V_2) = P_1^\varepsilon\left(\frac{1}{\sqrt{\varepsilon}} F_\varepsilon(V_1 + V_2)\right), \qquad g_\varepsilon(V_1, V_2) = P_2^\varepsilon\left(\frac{1}{\sqrt{\varepsilon}} F_\varepsilon(V_1 + V_2)\right) \tag{4.10}$$

for $V_1 \in H_1^\varepsilon$ and $V_2 \in H_2^\varepsilon$. Observe that this system has the form of (4.2).

We proceed now to the proof of Theorem 2.1 by proving that hypotheses (H1)–(H3) of Theorem 4.2 are fulfilled for system (4.9). This is done in the following Lemmas 4.6–4.8. We begin with the two easier ones.

Lemma 4.6 (*Hypothesis (H1)*). *For $V = (V_1, V_2) \in \mathcal{H}$, $h_\varepsilon(V)$ and $g_\varepsilon(V)$ defined in (4.10) and for $\varepsilon_0 > 0$ fixed, there exist positive constants L_h, N_h, L_g and N_g, all of them independent from ε if $0 < \varepsilon < \varepsilon_0$, such that for all V and $V' \in \mathcal{H}$ we have:*
(a) $\left\|h_\varepsilon(V) - h_\varepsilon(V')\right\|_\varepsilon \le L_h \left\|V - V'\right\|_\varepsilon$ *and* $\|h_\varepsilon(V)\|_\varepsilon \le N_h$;
(b) $\left\|g_\varepsilon(V) - g_\varepsilon(V')\right\|_\varepsilon \le L_g \left\|V - V'\right\|_\varepsilon$ *and* $\|g_\varepsilon(V)\|_\varepsilon \le N_g$.

Proof. In order to prove this lemma it suffices to show that $(1\sqrt{\varepsilon})F_\varepsilon$ is Lipschitz and bounded and that the projections P_1^ε and P_2^ε of elements of the form $\begin{pmatrix}(0,0)\\(a,a)\end{pmatrix}$, $a \in \mathbb{C}$, are bounded. And both parts need to be fulfilled in the $\|\cdot\|_\varepsilon$ norm and uniformly on ε if $\varepsilon < \varepsilon_0$. Observe that this suffices because $(1/\sqrt{\varepsilon})F_\varepsilon(V)$ has the previous form.

The first part comes from the fact that f is a Lipschitz and bounded function. For the second one, it is enough to see that P_2^ε (the finite dimensional projection) is bounded when it is applied to elements of the form given above. This can be proved by a calculation that essentially consists of comparing P_2^ε with the limit projection P_2^0. The details of these calculations can be found in [12]. □

Lemma 4.7 (*Hypothesis (H3)*). *The semigroup generated by the finite dimensional operator $\left(1/\sqrt{\varepsilon}\right) A_\varepsilon|_{H_2^\varepsilon}$ satisfies that*

$$\left\|\mathrm{e}^{(1/\sqrt{\varepsilon})A_\varepsilon|_{H_2^\varepsilon} t}\, V_2\right\|_\varepsilon \le \mathrm{e}^{\rho(\varepsilon)|t|}\, \|V_2\|_\varepsilon \tag{4.11}$$

for $\rho(\varepsilon) = 2\sqrt{2} + O(\sqrt{\varepsilon})$, for all $V_2 \in H_2^\varepsilon$ and $t \in \mathbb{R}$.

Proof. The details of the proof can also be found in [12]. The main idea is that the subspace H_2^ε is totally explicit (see (4.6) and (4.7)). This allows us to write

$$\sup_{0\neq V_2 \in H_2^\varepsilon} \frac{\left\|\left(\frac{1}{\sqrt{\varepsilon}} A_\varepsilon\right) V_2\right\|_\varepsilon^2}{\|V_2\|_\varepsilon^2}$$

explicitly and bound the terms that appear in this expression using their developments in powers of $\sqrt{\varepsilon}$. □

It only remains for (H2) to be proved. This hypothesis is the result given in the next lemma and turns out to be the most difficult hypothesis of Theorem 4.2 to check in our case (due to the non-self-adjointness of the operators). For proving this lemma the results of the Appendix are essential. Also, we use some of the definitions and notation of this Appendix that, for brevity, are not included in the present section.

Lemma 4.8 (*Hypothesis (H2)*). *The operator $(A_\varepsilon, \mathcal{D}(A_\varepsilon))$ for $\varepsilon > 0$ satisfies the following:*

(a) *A_ε is a relatively bounded perturbation of A_0 in the $\|\cdot\|_{\mathcal{H}}$ norm with an A_0-bound independent of $\varepsilon < \varepsilon_a$ for a certain $\varepsilon_a > 0$.*

(b) *There exist $\varepsilon_b > 0$ and a sector Σ centered in a certain $c > 0$ and with angle $\pi/2 + \delta_0'$, $\delta_0' \in (0, \pi/2]$, and there exists also a constant $M > 0$ (all of them independent of $\varepsilon < \varepsilon_b$) such that, if $\varepsilon < \varepsilon_b$,*

$$\overline{\Sigma} \subset \rho(A_\varepsilon) \quad \text{and} \quad \|R(\lambda, A_\varepsilon)\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \leq \frac{M}{|\lambda|} \quad \forall \lambda \in \overline{\Sigma}$$

where $\rho(A_\varepsilon)$ represents the resolvent set of A_ε.

(c) *There exists $\varepsilon_c > 0$ and $M' \geq 1$ (independent of $\varepsilon < \varepsilon_c$) such that*

$$\left\| \mathrm{e}^{(1/\sqrt{\varepsilon}) A_\varepsilon|_{H_1^\varepsilon} t}\, V_1 \right\|_\varepsilon \leq M'\, \mathrm{e}^{-\beta(\varepsilon)t}\, \|V_1\|_\varepsilon \quad \text{if } \varepsilon < \varepsilon_c \tag{4.12}$$

for all $V_1 \in H_1^\varepsilon$, where $\beta(\varepsilon) = c(\alpha)/\sqrt[4]{\varepsilon}$ and $0 < c(\alpha) < \min\{1/\alpha, \alpha\pi^2/2\}$.

Remark 4.9. Notice that the result given in (b) is in the $\|\cdot\|_{\mathcal{H}}$ norm, while in (c) the norm which is used is $\|\cdot\|_\varepsilon$.

The objective of Lemma 4.8 is part (c), that is the uniform bound of the semigroup generated by the operator projected in the infinite dimensional subspace H_1^ε. The fact that $\left(1/\sqrt{\varepsilon}\right) A_\varepsilon|_{H_1^\varepsilon}$ is a sectorial operator in H_1^ε gives automatically an inequality of the same type as (4.12) but with a constant $M(\varepsilon) \geq 1$, depending on ε, that may go to ∞ when $\varepsilon \to 0$. Observe that this does not happen with a self-adjoint operator because an inequality of type (4.12) is obtained with a constant $M = 1$. This is the case for the works of Fusco in [8], Carvalho and Lozada-Cruz in [5] and Carbone in [2], to cite just some. And this is the difficulty in proving Lemma 4.8 when the operator is not self-adjoint, as in our case.

The idea of the proof is the following. We know that when dealing with a sectorial operator, the constant M' that appears in (4.12) is related to the bound of the resolvent operator. Also, the exponent $\beta(\varepsilon)$ is related to the real part of the spectrum of the corresponding operator. As we will see, these constants can be given in terms of a sector included in the resolvent set and a certain bound for the resolvent operator in this sector. Therefore, we will try to find these constants for the resolvent of $A_\varepsilon|_{H_1^\varepsilon}$ uniformly on ε if ε is sufficiently small.

The way to do this is by using that $A_\varepsilon \to A_0$ in the generalized sense (see Theorem 3.2). This will imply that the corresponding eigenspaces are also close in some sense. Finally, this will allow us to somehow compare the sectors and resolvents of the corresponding projected operators. This part is done in the Appendix.

Proof. First we will show part (a), which is used in proving (b), which is used to prove (c).

(a) We have to see that

$$\|(A_\varepsilon - A_0)V\|_{\mathcal{H}} \leq a\, \|A_0\, V\|_{\mathcal{H}} + b\, \|V\|_{\mathcal{H}}, \quad V = \begin{pmatrix} (u, u(1)) \\ (v, v(1)) \end{pmatrix} \in \mathcal{D}(A_0)$$

with $a, b > 0$ being independent of ε if it is small enough (see [7] for definitions). By rewriting the previous norms, using Cauchy's inequality and after some calculations, we obtain that

$$\|(A_\varepsilon - A_0)V\|^2_{\mathcal{H}} \leq (\varepsilon a(\alpha))^2\, \|A_0\, V\|^2_{\mathcal{H}} + (\varepsilon b(r))^2\, \|V\|^2_{\mathcal{H}}$$

for

$$a^2(\alpha) = \max\left\{2(1 + c_1)\left(1 + \frac{1}{c_2}\right)\alpha^2,\ (1 + c_1)\right\}$$

$$b^2(r) = \max\left\{2(1 + c_1)\,(1 + c_2)\,, \left(1 + \frac{1}{c_1}\right) r^2\right\}$$

for any $c_1, c_2 > 0$ that we fix from now on. On fixing $\varepsilon_a > 0$ the relative bounds are $\varepsilon_a\, a(\alpha)$ and $\varepsilon_a\, b(r)$ if $\varepsilon < \varepsilon_a$, both independent of ε (see [12] for details).

(b) Observe that A_0 is a sectorial operator (see Theorem 3.1), and that there exist $\delta_0 \in (0, \pi/2]$ and $M_0 \geq 1$ such that

$$\overline{\Sigma}_{\pi/2+\delta_0} \setminus \{0\} = \{z \in \mathbb{C}, |\arg(z)| \leq \pi/2 + \delta_0\} \setminus \{0\} \subset \rho(A_0)$$

and

$$\|R(\lambda, A_0)\|_{\mathcal{H}} \leq \frac{M_0}{|\lambda|} \quad \forall\, 0 \neq \lambda \in \overline{\Sigma}_{\pi/2+\delta_0}.$$

Let us choose $\varepsilon_a > 0$ in (a) such that $a_0 = \varepsilon_a\, a(\alpha) < \frac{1}{2(1+M_0)}$. As ε_a, c_1 and c_2 have been fixed, so is $\varepsilon_a b(r)$. Therefore, Lemma A.3 and the Remark A.4 of the Appendix imply that

$$\overline{\Sigma}_{\pi/2+\delta_0'}(c_0) = \{z \in \mathbb{C}, |\arg(z - c_0)| \leq \pi/2 + \delta_0'\} \subset \rho(A_\varepsilon) \quad \text{if } \varepsilon < \varepsilon_a$$

and

$$\|R(\lambda, A_\varepsilon)\|_{\mathcal{H}} \leq \frac{2M_0}{|\lambda|} \quad \forall \lambda \in \overline{\Sigma}_{\pi/2+\delta_0'}(c_0), \text{ if } \varepsilon < \varepsilon_a \tag{4.13}$$

for $0 < \delta_0' \leq \delta_0$ and $c_0 = 2\, \frac{\varepsilon_a b(r)\, M_0}{1/2 - \varepsilon_a a(\alpha)(M_0+1)}$.

(c) We know that $\sigma(A_0)$ is separated into two parts by $\{\operatorname{Re} z = -c(\alpha)\}$, for $0 < c(\alpha) < \min\{1/\alpha, \alpha\pi^2/2\}$ with $\operatorname{Re}\sigma(A_0|_{H_1^0}) < -c(\alpha) < \operatorname{Re}\sigma(A_0|_{H_2^0}) \leq 0$ (see [13]).

We also know that A_ε tends to A_0 in the generalized sense in the $\|\cdot\|_{\mathcal{H}}$ norm when $\varepsilon \to 0$ (see Theorem 3.2). And, from part (b), we know that for $\varepsilon < \varepsilon_a$ the resolvent set of A_ε contains a sector independent of ε, in which the resolvent operator is bounded uniformly on ε in the form (4.13). Therefore, Theorem A.1 of the Appendix implies that there exists $\varepsilon_c > 0$ such that

$$\left\| \mathrm{e}^{A_\varepsilon|_{H_1^\varepsilon} t}\, V \right\|_{\mathcal{H}} \leq M\, \mathrm{e}^{-c(\alpha)\, t}\, \|V\|_{\mathcal{H}} \quad \forall V \in H_1^\varepsilon, \forall t \geq 0, \text{ if } \varepsilon < \varepsilon_c$$

for a certain M which is independent of ε. So we have that

$$\left\| \mathrm{e}^{(1/\sqrt{\varepsilon}) A_\varepsilon|_{H_1^\varepsilon} t}\, V \right\|_{\mathcal{H}} \leq M\, \mathrm{e}^{\frac{-c(\alpha)}{\sqrt{\varepsilon}}\, t}\, \|V\|_{\mathcal{H}} \quad \forall V \in H_1^\varepsilon, \forall t \geq 0 \tag{4.14}$$

if $\varepsilon < \varepsilon_c$. We want to show an inequality of this kind, but with the $\|\cdot\|_\varepsilon$ norm.

As A_ε is a dissipative operator (see Theorem 3.1), it is true that

$$\left\| \mathrm{e}^{(1/\sqrt{\varepsilon}) A_\varepsilon|_{H_1^\varepsilon} t}\, V \right\|_{\varepsilon} \leq \|V\|_\varepsilon\ \forall t \geq 0, \quad \forall V \in H_1^\varepsilon. \tag{4.15}$$

On the other hand and using the relation between the equivalent norms $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_\varepsilon$, (4.14) can be written as

$$\left\| \mathrm{e}^{(1/\sqrt{\varepsilon}) A_\varepsilon|_{H_1^\varepsilon} t}\, V \right\|_{\varepsilon} \leq \frac{M}{\sqrt{\varepsilon}}\, \mathrm{e}^{\frac{-c(\alpha)}{\sqrt{\varepsilon}}\, t}\, \|V\|_\varepsilon \quad \forall V \in H_1^\varepsilon, \forall t \geq 0, \text{ if } \varepsilon < \varepsilon_c. \tag{4.16}$$

Joining (4.15) and (4.16), we have the following bound:

$$\left\| \mathrm{e}^{(1/\sqrt{\varepsilon}) A_\varepsilon|_{H_1^\varepsilon} t} \right\|_{\mathcal{L}(\mathcal{H},\mathcal{H};\, \varepsilon)} \leq \min\left\{ \frac{1}{\sqrt{\varepsilon}}\, M\, \mathrm{e}^{\frac{-c(\alpha)}{\sqrt{\varepsilon}}\, t}, 1 \right\}, \quad \forall t \geq 0, \text{ if } \varepsilon < \varepsilon_c$$

for all $\varepsilon < \varepsilon_c$, where $\mathcal{L}(\mathcal{H}, \mathcal{H};\, \varepsilon)$ stands for the norm in $\mathcal{L}(\mathcal{H}, \mathcal{H})$ induced by the $\|\cdot\|_\varepsilon$ norm. Multiplying this inequality by $\mathrm{e}^{\frac{c(\alpha)}{\sqrt[4]{\varepsilon}}\, t}$ we can check that

$$\left\| \mathrm{e}^{(1/\sqrt{\varepsilon}) A_\varepsilon|_{H_1^\varepsilon} t} \right\|_{\mathcal{L}(\mathcal{H},\mathcal{H};\, \varepsilon)} \mathrm{e}^{\frac{c(\alpha)}{\sqrt[4]{\varepsilon}}\, t} \leq \min\left\{ \frac{1}{\sqrt{\varepsilon}}\, M\, \mathrm{e}^{\left(\frac{-c(\alpha)}{\sqrt{\varepsilon}} + \frac{c(\alpha)}{\sqrt[4]{\varepsilon}}\right) t}, \mathrm{e}^{\frac{c(\alpha)}{\sqrt[4]{\varepsilon}}\, t} \right\} \leq M'(\varepsilon)$$

for all $t \geq 0$ if $\varepsilon < \varepsilon_c$, where

$$M'(\varepsilon) = \left(\frac{M}{\sqrt{\varepsilon}}\right)^{\sqrt[4]{\varepsilon}}.$$

Observe that $\lim_{\varepsilon\to 0} M'(\varepsilon) = 1$, so we can say that this constant $M'(\varepsilon)$ is bounded by a certain $M' > 0$, a constant independent of ε if $\varepsilon < \varepsilon_c$. So, we conclude that

$$\left\| \mathrm{e}^{(1/\sqrt{\varepsilon})A_\varepsilon|_{H_1^\varepsilon} t} \right\|_{\mathcal{L}(\mathcal{H},\mathcal{H};\,\varepsilon)} \leq M' \mathrm{e}^{\frac{-c(\alpha)}{\sqrt[4]{\varepsilon}} t} \quad \forall t \geq 0, \text{if } \varepsilon < \varepsilon_c$$

with M' independent of ε. With this affirmation, we have finished this proof. □

As all the hypotheses have been shown, the existence of an exponentially attracting invariant manifold for system (4.9) is almost proved.

Proof of Theorem 2.1. In the previous Lemmas 4.6–4.8 the Hypotheses (H1)–(H3) of Theorem 4.2 for the system (4.9) have been proved. So it only remains to check whether the exponents $\rho(\varepsilon)$ and $\beta(\varepsilon)$ of Lemmas 4.7 and 4.8 satisfy that

$$\lim_{\varepsilon\to 0} (\beta(\varepsilon) - \rho(\varepsilon)) = +\infty.$$

This is true as $\rho(\varepsilon) = 2\sqrt{2} + O(\sqrt{\varepsilon})$ and $\beta(\varepsilon) = c(\alpha)/\sqrt[4]{\varepsilon}$. Hence, Theorem 4.2 is satisfied for our problem and, therefore, Theorem 2.1 is proved. □

Remark 4.10. Notice that the norm in which the invariant manifold and its derivative tend to 0 is the $\|\cdot\|_\varepsilon$ norm, which is constantly changing when $\varepsilon \to 0$. This norm will be essential in Section 5 below for obtaining the explicit limit ODE. However, Theorem 2.1 is also true in the $\|\cdot\|_{\mathcal{H}}$ norm, so there also exists an exponentially attracting invariant manifold tending to 0 in the $\mathcal{C}^1$ topology with this norm. The problem is that this convergence in the $\|\cdot\|_{\mathcal{H}}$ norm is not enough to prove that system (4.9) tends to the limit Eq. (2.4) in the $\mathcal{C}^1$ norm unless $f \equiv f(u(1))$. But in that case, the limit equation would not be structurally stable and, hence, the two dynamics could not be compared. So Theorem 2.1 in the $\|\cdot\|_\varepsilon$ norm, not just in the $\|\cdot\|_{\mathcal{H}}$ norm, is necessary.

5. Consequences: The explicit limit ODE

We first see some consequences of Theorem 2.1. In Theorem 2.1 we have proved the existence of a family of exponentially attracting invariant manifolds given by $\eta_\varepsilon : H_2^\varepsilon \to H_1^\varepsilon$ satisfying that

$$\lim_{\varepsilon\to 0} \sup_{V_2\in H_2^\varepsilon} \|\eta_\varepsilon(V_2)\|_\varepsilon = 0 \tag{5.1}$$

and

$$\lim_{\varepsilon\to 0} \sup_{V_2\in H_2^\varepsilon} \|D\eta_\varepsilon(V_2)\|_{\mathcal{L}(H_2^\varepsilon, H_1^\varepsilon;\varepsilon)} = \lim_{\varepsilon\to 0} \sup_{V_2\in H_2^\varepsilon} \left(\sup_{W_2\in H_2^\varepsilon} \frac{\|D\eta_\varepsilon(V_2)[W_2]\,\|_\varepsilon}{\|W_2\|_\varepsilon} \right) = 0. \tag{5.2}$$

We can write (5.1) and (5.2) in more detail. First of all, as H_2^ε is a two-dimensional space, we can think of η_ε as a function of the two coordinates of an element of H_2^ε in a certain basis. Then, we can write η_ε as

$$\eta_\varepsilon(a,b) = \begin{pmatrix} (\eta_\varepsilon^{11}(a,b)\ ,\ \eta_\varepsilon^{12}(a,b)) \\ (\eta_\varepsilon^{21}(a,b)\ ,\ \eta_\varepsilon^{22}(a,b)) \end{pmatrix} \in H_1^\varepsilon \subset \mathcal{H}$$

for $a, b \in \mathbb{C}$, and with $\eta_\varepsilon^{12}(\cdot,\cdot) = \eta_\varepsilon^{11}(\cdot,\cdot)|_{x=1}$ because $\eta_\varepsilon \in \mathcal{H}$ (see [12]).

In particular, by writing the expression (5.1) for such elements, we obtain

$$\lim_{\varepsilon\to 0} \sup_{a,b\in\mathbb{C}} \frac{|\eta_\varepsilon^{22}(a,b)|}{\sqrt{\varepsilon}} = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \sup_{a,b\in\mathbb{C}} |\eta_\varepsilon^{12}(a,b)| = 0 \tag{5.3}$$

(as η_ε^{12} is the value of η_ε^{11} at the end $x = 1$). The same idea applied to $D\eta_\varepsilon$ and the limit (5.2) takes us to the conclusion that

$$\lim_{\varepsilon\to 0}\sup_{a,b\in\mathbb{C}} \frac{\left|(\partial_1\eta_\varepsilon^{22})(a,b)\right|}{\sqrt{\varepsilon}} = 0, \qquad \lim_{\varepsilon\to 0}\sup_{a,b\in\mathbb{C}} \left|(\partial_1\eta_\varepsilon^{12})(a,b)\right| = 0 \tag{5.4}$$

and that

$$\lim_{\varepsilon\to 0}\sup_{a,b\in\mathbb{C}} \left|(\partial_2\eta_\varepsilon^{22})(a,b)\right| = 0, \qquad \lim_{\varepsilon\to 0}\sup_{a,b\in\mathbb{C}} \sqrt{\varepsilon}\left|(\partial_2\eta_\varepsilon^{12})(a,b)\right| = 0. \tag{5.5}$$

Of course, a more detailed calculation is needed in these cases (see [12]).

Now we proceed to compute the limit ODE. We have seen in Theorem 2.1 that, for large times and ε small enough, solutions of the re-scaled Eq. (4.4) tend to solutions of Eq. (2.3). These ones lie on S_ε, which tends to be a flat manifold when $\varepsilon \to 0$, in the $\mathcal{C}^1$ topology. As we said, this important contribution of Theorem 4.2 will allow us to write the limit ODE explicitly. This will be done in this section.

Observe that if A_ε was a self-adjoint operator, the projection onto the eigenspaces would be easily obtained. Therefore, we could find explicitly the equation on the invariant manifold and, hence, its limit when $\varepsilon \to 0$ could be calculated. But the fact of A_ε being a non-self-adjoint operator makes it difficult to obtain these projections explicitly and so also an explicit expression for the equation on S_ε.

In order to obtain this expression it will be crucial again to have the generalized convergence of $A_\varepsilon \to A_0$ when $\varepsilon \to 0$ (see Theorem 3.2). This convergence implies that the spaces H_2^ε and H_2^0, defined in (4.5) and (4.8), are isomorphic and their corresponding projections converge in norm when ε is small enough. Somehow, this will provide us with a manifold $S_0 = V_0 + \eta_\varepsilon(V_0)$, $V_0 \in H_2^0$, which we do know how to project on, and that is isomorphic to $S_\varepsilon = V_2 + \eta_\varepsilon(V_2)$, $V_2 \in H_2^\varepsilon$, and close to it when ε is small enough. Moreover, as these spaces and projections are explicit, the equation on S_0 (equivalent to the one on S_ε) and its limit will be computed explicitly.

In this sense the techniques developed in this section for finding the explicit limit ODE represent another novelty with respect to works such as those mentioned in Section 4, in which the operators are self-adjoint.

Let us see all these ideas in more detail. For this section, let us define $A_N = \left(1/\sqrt{\varepsilon}\right) A_\varepsilon$ and $F_N = \left(1/\sqrt{\varepsilon}\right) F_\varepsilon$.

Let now call P_0 the restriction of P_2^0 (the projection onto H_2^0) to H_2^ε, that is,

$$P_0 = P_2^0|_{H_2^\varepsilon} : H_2^\varepsilon \subset \mathcal{H} \longrightarrow H_2^0$$

and

$$Q_\varepsilon : H_2^0 \longrightarrow H_2^\varepsilon$$
$$\begin{pmatrix}(u(1)x, u(1))\\ (v(1)x, v(1))\end{pmatrix} \longrightarrow a\,\Psi_\varepsilon^1 + b\,\Psi_\varepsilon^2$$

where $u(1), v(1) \in \mathbb{C}$ are the coordinates of an element of H_2^0 and $a, b \in \mathbb{C}$ are such that $P_0\, Q_\varepsilon = Id|_{H_2^0}$, with Ψ_ε^1, Ψ_ε^2 being defined in (4.6) and (4.7). These coefficients a and b can be obtained explicitly (see [12]).

Observe that $P_0 = Q_\varepsilon^{-1}$ is an isomorphism between H_2^ε and H_2^0. This isomorphism allows a change of variables in (2.3): $V_2 = Q_\varepsilon V_0$, $V_0 \in H_2^0$. With this change of variables and applying $P_0 = Q_\varepsilon^{-1}$ to the whole Eq. (2.3), this is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t} V_0 = (P_0 A_N Q_\varepsilon)\, V_0 + P_0 P_2^\varepsilon \left[F_N\left(Q_\varepsilon V_0 + \eta_\varepsilon(Q_\varepsilon V_0)\right)\right], \qquad V_0 \in H_2^0. \tag{5.6}$$

This is the same equation, but applied to elements of H_2^0 instead of elements of H_2^ε. This will make the equation easier to write. In order to make it still easier, it is convenient to apply (5.6) to elements of the form

$$V_0 = \begin{pmatrix}(u(1)x, u(1))\\ (v(1)x, v(1))\end{pmatrix} = \begin{pmatrix}(u(1)x, u(1))\\ (\sqrt{\varepsilon}w(1)x, \sqrt{\varepsilon}w(1))\end{pmatrix}. \tag{5.7}$$

We can think of this as another change of variables in order to capture the slow moving solutions. Also remember from the first part of this section that $\eta_\varepsilon(Q_\varepsilon V_0)$ can be written in terms of the coordinates of $Q_\varepsilon V_0 \in H_2^\varepsilon$. In this case, this means that

$$\eta_\varepsilon(Q_\varepsilon V_0) = \eta_\varepsilon(u(1), v(1)) = \eta_\varepsilon(u(1), \sqrt{\varepsilon} w(1)).$$

So, summarizing, the equation that we are going to write explicitly to obtain its limit is Eq. (5.6), where V_0 has the form (5.7). But observe that, by adding and subtracting terms, this equation can be seen to be equivalent to the equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} V_0 &= (P_0 A_N Q_\varepsilon) V_0 + P_0 P_2^\varepsilon \left[F_N\left(Q_\varepsilon V_0 + \eta_\varepsilon(Q_\varepsilon V_0)\right)\right] - P_2^0 P_2^0 \left[F_N\left(Q_\varepsilon V_0 + \eta_\varepsilon(Q_\varepsilon V_0)\right)\right] \\ &\quad + P_2^0 P_2^0 \left[F_N\left(Q_\varepsilon V_0 + \eta_\varepsilon(Q_\varepsilon V_0)\right)\right] - P_2^0 F_N(V_0) + P_2^0 F_N(V_0). \end{aligned} \tag{5.8}$$

Essentially, the lemmas below prove that the difference terms of (5.8) tend to zero in the $\mathcal{C}^1$ norm when $\varepsilon \to 0$, while the rest of the terms tend to the limit Eq. (2.4) given in Theorem 2.2 when $\varepsilon \to 0$, also in the $\mathcal{C}^1$ topology.

Lemma 5.1 (*Linear Operator*). *Let V_0 be of the form given in (5.7). Then*

$$(P_0 A_N Q_\varepsilon) V_0 = w(1) \begin{pmatrix} (x,1) \\ (0,0) \end{pmatrix} - \sqrt{\varepsilon}\left[c_1(\varepsilon) u(1) + (c_2(\varepsilon) + r)\sqrt{\varepsilon} w(1)\right] \begin{pmatrix} (0,0) \\ (x,1) \end{pmatrix} \tag{5.9}$$

where it can be seen that

$$\lim_{\varepsilon \to 0} c_1(\varepsilon) = 1, \qquad \lim_{\varepsilon \to 0} c_2(\varepsilon) = \alpha.$$

Proof. This result is proved by writing explicitly each term of $(P_0 A_N Q_\varepsilon) V_0$ using the definitions of each operator, developing the expressions of $c_1(\varepsilon)$ and $c_2(\varepsilon)$ as a power series of ε and taking limits when $\varepsilon \to 0$ (see [12] for details). □

Lemma 5.2 (*First-Difference Term*). *If we define*

$$P_2^\varepsilon \begin{pmatrix} (0,0) \\ (-f(\xi,\eta), -f(\xi,\eta)) \end{pmatrix} = \begin{pmatrix} (p_\varepsilon^{11}(\xi,\eta), p_\varepsilon^{12}(\xi,\eta)) \\ (p_\varepsilon^{21}(\xi,\eta), p_\varepsilon^{22}(\xi,\eta)) \end{pmatrix} \in H_2^\varepsilon$$

then we have

$$\begin{aligned} &P_0 P_2^\varepsilon \left[F_N\left(Q_\varepsilon V_0 + \eta_\varepsilon(Q_\varepsilon V_0)\right)\right] - P_2^0 P_2^0 \left[F_N\left(Q_\varepsilon V_0 + \eta_\varepsilon(Q_\varepsilon V_0)\right)\right] \\ &\quad = \sqrt{\varepsilon} \begin{pmatrix} \left(p_\varepsilon^{12}(\tilde{u},\tilde{w})\, x\,,\ p_\varepsilon^{12}(\tilde{u},\tilde{w})\right) \\ \left(\left[p_\varepsilon^{22}(\tilde{u},\tilde{w}) + f(\tilde{u},\tilde{w})\right] x, \left[p_\varepsilon^{22}(\tilde{u},\tilde{w}) + f(\tilde{u},\tilde{w})\right]\right) \end{pmatrix} \end{aligned}$$

where

$$\tilde{u} = u(1) + \eta_\varepsilon^{12}(u(1), \sqrt{\varepsilon} w(1)), \qquad \tilde{w} = w(1) + \frac{1}{\sqrt{\varepsilon}} \eta_\varepsilon^{22}(u(1), \sqrt{\varepsilon} w(1)).$$

Moreover, the following limits are true:

(a) $\lim_{\varepsilon \to 0}\left(\sup_{u(1),\, w(1) \in \mathbb{C}} |p_\varepsilon^{12}(\tilde{u},\tilde{w})|\right) = 0$ *and*

$$\lim_{\varepsilon \to 0}\left(\sup_{u(1), w(1) \in \mathbb{C}} \left|\frac{\partial}{\partial u(1)}\left[p_\varepsilon^{12}(\tilde{u},\tilde{w})\right]\right|\right) = 0$$

$$\lim_{\varepsilon \to 0}\left(\sup_{u(1), w(1) \in \mathbb{C}} \left|\frac{\partial}{\partial w(1)}\left[p_\varepsilon^{12}(\tilde{u},\tilde{w})\right]\right|\right) = 0.$$

(b) $\lim_{\varepsilon \to 0}\left(\sup_{u(1),\, w(1) \in \mathbb{C}} |p_\varepsilon^{22}(\tilde{u},\tilde{w}) + f(\tilde{u},\tilde{w})|\right) = 0$ *and*

$$\lim_{\varepsilon \to 0}\left(\sup_{u(1),\, w(1) \in \mathbb{C}} \left|\frac{\partial\left[p_\varepsilon^{22}(\tilde{u},\tilde{w}) + f(\tilde{u},\tilde{w})\right]}{\partial u(1)}\right|\right) = 0$$

$$\lim_{\varepsilon \to 0}\left(\sup_{u(1), w(1) \in \mathbb{C}} \left|\frac{\partial\left[p_\varepsilon^{22}(\tilde{u},\tilde{w}) + f(\tilde{u},\tilde{w})\right]}{\partial w(1)}\right|\right) = 0.$$

Proof. Again, the details of the proof can be found in [12]. The main idea is the following. Let G denote the following operator:

$$G\begin{pmatrix}(u,u(1))\\(v,\beta)\end{pmatrix}=G(u(1),\beta)=\begin{pmatrix}(0,0)\\(f(u(1),\beta),f(u(1),\beta))\end{pmatrix}.$$

Observe that

$$P_2^0(P_2^\varepsilon-P_2^0)\,F_N\;(Q_\varepsilon V_0+\eta_\varepsilon(Q_\varepsilon V_0))=\sqrt{\varepsilon}P_2^0(P_2^\varepsilon-P_2^0)\,G\,(\tilde{u},\tilde{w})$$

where $\tilde{u}$, $\tilde{w}$ are the functions given in the statement of the lemma. By writing the explicit expressions for the operators involved in this term we obtain

$$P_2^0(P_2^\varepsilon-P_2^0)\,G\,(\tilde{u},\tilde{w})=\begin{pmatrix}\left(p_\varepsilon^{12}(\tilde{u},\tilde{w})\,x,\;p_\varepsilon^{12}(\tilde{u},\tilde{w})\right)\\\left(\left[p_\varepsilon^{22}(\tilde{u},\tilde{w})+f(\tilde{u},\tilde{w})\right]x,\left[p_\varepsilon^{22}(\tilde{u},\tilde{w})+f(\tilde{u},\tilde{w})\right]\right)\end{pmatrix}.$$

For the second part we see that the norm of this operator together with the norm of its derivatives tend to 0 when $\varepsilon\to 0$.

This is true due to the fact that $\|P_2^\varepsilon-P_2^0\|_{\mathcal{L}(\mathcal{H},\mathcal{H})}\to 0$ when $\varepsilon\to 0$ (see Remark 4.5) and to the fact that $\|G\|_{\mathcal{C}^0_{\mathcal{H}}}=\sup_{V\in\mathcal{H}}\|G(V)\|_{\mathcal{H}}$ is bounded independently of ε. So we have that

$$sup_{u(1),w(1)\in\mathbb{C}}\left\|P_2^0(P_2^\varepsilon-P_2^0)\,G\,(\tilde{u},\tilde{w})\right\|_{\mathcal{H}}\longrightarrow_{\varepsilon\to 0}0.$$

Developing the expression for this norm we obtain the first limits of point (a) and point (b) of the present lemma.

Applying the same idea to the differential operator $D\left[P_2^0(P_2^\varepsilon-P_2^0)\,G\right]$ and using the inequalities (5.4), (5.5), we can conclude the other limits. □

Lemma 5.3 (*Second-Difference Term*). *The second-difference term of* (5.8) *is given by*

$$\begin{aligned}&P_2^0P_2^0\left[F_N\left(Q_\varepsilon V_0+\eta_\varepsilon(Q_\varepsilon V_0)\right)\right]-P_2^0F_N(V_0)\\&\quad=\sqrt{\varepsilon}\begin{pmatrix}(0,0)\\\left(-x\left[f(\tilde{u},\tilde{w})-f(u(1),w(1))\right],-\left[f(\tilde{u},\tilde{w})-f(u(1),w(1))\right]\right)\end{pmatrix},\end{aligned}$$

where

$$\tilde{u}=u(1)+\eta_\varepsilon^{12}(u(1),\sqrt{\varepsilon}w(1)),\qquad \tilde{w}=w(1)+\frac{1}{\sqrt{\varepsilon}}\eta_\varepsilon^{22}(u(1),\sqrt{\varepsilon}w(1)).$$

We claim that the following limits are true:

$$\lim_{\varepsilon\to 0}\left(\sup_{u(1),w(1)\in\mathbb{C}}|f(\tilde{u},\tilde{w})-f(u(1),w(1))|\right)=0$$

$$\lim_{\varepsilon\to 0}\left(\sup_{u(1),w(1)\in\mathbb{C}}\left|\frac{\partial}{\partial u(1)}\left[f(\tilde{u},\tilde{w})-f(u(1),w(1))\right]\right|\right)=0$$

$$\lim_{\varepsilon\to 0}\left(\sup_{u(1),w(1)\in\mathbb{C}}\left|\frac{\partial}{\partial w(1)}\left[f(\tilde{u},\tilde{w})-f(u(1),w(1))\right]\right|\right)=0.$$

Proof. Writing these expressions in detail, it can be seen that the second-difference term can be written as given in the statement of this lemma.

The fact that f is Lipschitz, the definitions of $\tilde{u}$, $\tilde{w}$ and the limits of the components of η_ε given in (5.3) (because $\eta_\varepsilon\to 0$ in the $\mathcal{C}^1$ topology in $\|\cdot\|_\varepsilon$) imply the first limit of the lemma. The second and the third ones come from the regularity of f, the definitions of $\tilde{u}$, $\tilde{w}$ and the limits (5.4)–(5.5) (also due to the convergence to 0 of η_ε in the $\mathcal{C}^1$ topology in $\|\cdot\|_\varepsilon$).

Remark 5.4. In order to prove the limits given in Lemmas 5.2 and 5.3, we need the results on the limits of the components of η_ε given from (5.3)–(5.5). The powers of ε that appear there are exactly the ones needed in computing these limits now. And these powers are the result of the convergence of η_ε to 0 in the $\mathcal{C}^1$ topology and also in the $\|\cdot\|_\varepsilon$ norm, and not only in the $\mathcal{C}^0$ topology or the $\|\cdot\|_{\mathcal{H}}$ norm.

With the results given in these previous three lemmas, the proof of Theorem 2.2 is almost complete:

Proof of Theorem 2.2. Using the results of Lemmas 5.1–5.3, Eq. (5.8) can be written as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\,u(1) = w(1) + \sqrt{\varepsilon}\,p_\varepsilon^{12}(\tilde u, \tilde w) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\,\sqrt{\varepsilon}w(1) = -\sqrt{\varepsilon}\,\left[c_1(\varepsilon)u(1) + (c_2(\varepsilon)+r)\,\sqrt{\varepsilon}w(1)\right] \\ +\sqrt{\varepsilon}\,\left[p_\varepsilon^{22}(\tilde u, \tilde w) + f(\tilde u, \tilde w)\right] \\ +\sqrt{\varepsilon}\,\left[-f(\tilde u, \tilde w) + f(u(1), w(1))\right] \\ -\sqrt{\varepsilon}\,f(u(1), w(1)). \end{cases} \tag{5.10}$$

Using again the results of these lemmas, we have for the first equation that

$$|w(1) + \sqrt{\varepsilon}\,p_\varepsilon^{12}(\tilde u, \tilde w)| \xrightarrow[\varepsilon\to 0]{\mathcal{C}^1} w(1).$$

And, for the second equation, observe that $\sqrt{\varepsilon}$ cancels in both sides of the equation and that:

(a) from Lemma 5.1 we obtain that $\left[c_1(\varepsilon)u(1) + (c_2(\varepsilon)+r)\,\sqrt{\varepsilon}w(1)\right] \xrightarrow[\varepsilon\to 0]{\mathcal{C}^1} u(1)$,

(b) from the second part of Lemma 5.2 we have that $\left|\left[p_\varepsilon^{22}(\tilde u, \tilde w) + f(\tilde u, \tilde w)\right]\right| \xrightarrow[\varepsilon\to 0]{\mathcal{C}^1} 0$, and

(c) from Lemma 5.3 we can say that $|-f(\tilde u, \tilde w) + f(u(1), w(1))| \xrightarrow[\varepsilon\to 0]{\mathcal{C}^1} 0$.

Combining all these results we obtain that the system of ODE's (5.10) converges in the $\mathcal{C}^1$ sense to the system (2.4).

Writing this system as a second-order equation we conclude that the solutions of the nonlinear model (1.3) with a convenient re-scaled time tend to the nonlinear second-order ODE (2.5) when $\varepsilon \to 0$ and $t \to \infty$.

Remark 5.5. As we said in Section 2, if we want to compare the asymptotic dynamics it has to be emphasized that the convergence is in the $\mathcal{C}^1$ topology. And also, it is absolutely crucial that this happens for the $\|\cdot\|_\varepsilon$ norm: if Theorem 2.1 was obtained only in $\|\cdot\|_{\mathcal{H}}$, we could only be assured of the convergence to the limit ODE in the $\mathcal{C}^0$ topology.

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Appendix. Bounds for restrictions of semigroups

In this section we only include the definitions and theorems directly used in Section 4 to prove the Hypothesis (H2) (Lemma 4.8). It should be said that, for brevity, all the previous tools needed in these definitions and theorems as well as the proof of the subsequent results are not included in the present paper and can be found in [12].

We denote the sector centered in $c \in \mathbb{R}$ and with angle $\delta \in (0, \pi]$ by

$$\Sigma_\delta(c) := \{\lambda \in \mathbb{C};\ |\arg(\lambda - c)| < \delta\}.$$

The next theorem and, more concretely, the third point of it have been used in the proof of Lemma 4.8 above.

Theorem A.1 (*Uniform Bound for the Semigroups Generated by the Restrictions*). *Let* $(T_0, \mathcal{D}(T_0))$ *be a sectorial operator in a Banach space* X *with* $\operatorname{Re}\sigma(T_0) \le 0$, *with the spectrum separated into two parts* σ_0^1, σ_0^2 *by* $\{\operatorname{Re} z = c_0\}$, *for a certain* $c_0 < 0$*. Let* $X = X_0 \oplus Y_0$ *be the associated decomposition of the whole space with* $\operatorname{Re}\sigma(T_0|_{X_0}) < c_0 < \operatorname{Re}\sigma(T_0|_{Y_0})$*. In this case, it is known that* $T_0|_{X_0}$ *generates an analytical semigroup and, therefore, there exists a sector centered in 0,* $\Sigma_0 = \Sigma_{\pi/2+\delta_0}(0)$*,* $\delta_0 \in (0, \pi/2]$*, with* $\overline{\Sigma}_0 \subset \rho(T_0|_{X_0})$ *and there exists* $C_0 \geq 1$ *such that*

$$\|R(\lambda, T_0|_{X_0})\|_{\mathcal{L}(X_0,X)} \leq \frac{C_0}{|\lambda|} \quad \forall\, 0 \neq \lambda \in \overline{\Sigma}_0.$$

Let $\{(T_\varepsilon, \mathcal{D}(T_\varepsilon))\}_{\varepsilon>0}$ *be a family of sectorial operators in* X*. Suppose that* $Re\,\sigma(T_\varepsilon) < 0$ *for all* $\varepsilon > 0$ *and that:*

(i) T_ε *tends to* T_0 *in the generalized sense when* $\varepsilon \to 0$*.*
(ii) *There exists a sector* $\Sigma = \Sigma_{\pi/2+\delta}(r)$*,* $\delta \in (0, \pi/2]$ *and* $r > 0$ *(independent of* ε*), and there exists* $C_\delta \geq 1$ *(also independent of* ε*) such that*

$$\overline{\Sigma} \subset \rho(T_\varepsilon) \quad \text{if } \varepsilon < \varepsilon_1 \quad \textit{and} \quad \|R(\lambda, T_\varepsilon)\|_{\mathcal{L}(X,X)} \leq \frac{C_\delta}{|\lambda|} \quad \forall\, \lambda \in \overline{\Sigma}$$

if $\varepsilon < \varepsilon_1$ *for a certain* $\varepsilon_1 > 0$*.*

Then, we can be sure that there exists $\varepsilon_0 > 0$ *such that if* $\varepsilon < \varepsilon_0$ *then:*

(a) *The spectrum of* T_ε *is separated into two parts* σ_ε^1*,* σ_ε^2 *by* $\{Re\, z = c_0\}$*, with* $X = X_\varepsilon \oplus Y_\varepsilon$ *being the associated decomposition of the whole space, where* $X_\varepsilon \cong X_0$ *and* $Y_\varepsilon \cong Y_0$*.*
(b) *There exists an angle* $\delta' \in (0, \delta)$ *and a constant* $C_{\delta'} \geq 1$*, both independent of* ε*, such that if* $\varepsilon < \varepsilon_0$ *the sector* $\Sigma' = \Sigma_{\pi/2+\delta'}(c_0)$ *satisfies*

$$\overline{\Sigma'} \setminus \{c_0\} \subset \rho(T_\varepsilon|_{X_\varepsilon}) \quad \textit{and} \quad \|R(\lambda, T_\varepsilon|_{X_\varepsilon})\|_{\mathcal{L}(X_\varepsilon,X_\varepsilon)} \leq \frac{C_{\delta'}}{|\lambda - c_0|} \quad \forall\, \lambda \in \overline{\Sigma'} \setminus \{c_0\}.$$

(c) *Finally, under these hypotheses we can be sure that*

$$\|e^{T_\varepsilon|_{X_\varepsilon} t}\|_{\mathcal{L}(X_\varepsilon,X_\varepsilon)} \leq M e^{c_0 t} \quad \textit{if } \varepsilon < \varepsilon_0$$

where $M = M(\Sigma', C_{\delta'})$ *and, therefore, it does not depend on* ε *if* $\varepsilon < \varepsilon_0$*.*

Remark A.2. The fact of $T_\varepsilon|_{X_\varepsilon}$ being analytic for each ε allows us to say that bounds (b) and (c) of the theorem are fulfilled but, possibly, with $C_{\delta'} = C(\varepsilon)$ and $M = M(\varepsilon)$. The interest of this result lies in the fact that these constants can be taken independent of ε if it is small enough. This fact is not obvious at all for a non-self-adjoint operator.

Also, hypothesis (ii) is necessary because the fact of T_ε being sectorial implies the existence of a sector $\Sigma_\varepsilon = \Sigma_{\pi/2+\delta_\varepsilon}(r_\varepsilon)$ and a constant $C_\varepsilon \geq 1$ such that $\overline{\Sigma}_\varepsilon \subset \rho(T_\varepsilon)$ and

$$\|R(\lambda, T_\varepsilon)\|_{\mathcal{L}(X,X)} \leq \frac{C_\varepsilon}{|\lambda - r_\varepsilon|} \quad \forall r_\varepsilon \neq \lambda \in \overline{\Sigma}_\varepsilon.$$

But in hypothesis (ii) one requires a sector and a constant being uniform on ε, at least if ε is small enough.

As we said, this theorem can be found in [12]. The idea behind the proof of this theorem is the following. As $T_\varepsilon \to T_0$ when $\varepsilon \to 0$ in the generalized sense (and, hence, $X_\varepsilon \cong X_0$ if ε is small enough) we expect that $T_\varepsilon|_{X_\varepsilon} \to T_0|_{X_0}$ in some sense or that $\|R(\lambda, T_\varepsilon|_{X_\varepsilon}) - R(\lambda, T_0|_{X_0})\| \to 0$ (we expect close operators restricted to close subspaces to have close resolvents). If this were true, the sector and constant for the resolvent of $T_0|_{X_0}$ (of course independent from ε) could also be used as the sector and constant for $R(\lambda, T_\varepsilon|_{X_\varepsilon})$. The problem is that $T_\varepsilon|_{X_\varepsilon}$ and $T_0|_{X_0}$ are not defined on the same spaces. So, to obtain the sector and constant for $T_\varepsilon|_{X_\varepsilon}$ uniformly on ε the hypotheses of Theorem A.1 are needed.

This is the main theorem that is used, but we will also need an adaptation of a result of Chapter III.2 of Engel and Nagel [7], that will allow us to prove, in Lemma 4.8, that the operator A_ε fulfills the hypothesis (ii) of Theorem A.1.

Lemma A.3 (*[7]*). *Let* $(A, \mathcal{D}(A))$ *be a closed operator such that its resolvent operator exists for all* $0 \neq \lambda \in \overline{\Sigma}_\delta(0)$ *and satisfies that*

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|}$$

for certain $\delta \geq 0$ *and* $M \geq 1$. *Suppose that* $(B, \mathcal{D}(B))$ *is an* A*-bounded operator with* A*-bound*

$$a < \frac{1}{M+1}.$$

Then, there exist constants $r \geq 0$ *and* $\widetilde{M} \geq 1$ *such that*

$$\overline{\Sigma}_\delta \cap \{z \in \mathbb{C}; |z| > r\} \subset \rho(A+B) \quad \textit{and} \quad \|R(\lambda, A+B)\| \leq \frac{\widetilde{M}}{|\lambda|}$$

for all $\lambda \in \overline{\Sigma}_\delta \cap \{z \in \mathbb{C}; |z| > r\}$.

Remark A.4. Revising the proof of this lemma in [7], we can be more precise with the constants in order to note what they depend on. This calculation can be found in [12]. There we conclude that, under the same hypothesis as in Lemma A.3 but with an A-bound satisfying that $a < \frac{1/2}{M+1}$, the conclusions of the previous lemma can be rewritten as

$$\overline{\Sigma}_{\delta'}(2r) \subset \rho(A+B) \quad \text{and} \quad \|R(\lambda, A+B)\| \leq \frac{2M}{|\lambda|}$$

for all $\lambda \in \overline{\Sigma}_{\delta'}(2r)$, for all $0 \leq \delta' \leq \delta$ and for r defined as

$$r = \frac{bM}{\frac{1}{2} - a(M+1)}$$

where M is the constant given in the hypothesis of the lemma.

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