Selfadjointness
and
optimal decay rates
in overdamped equations

Marta Pellicer (U. de Girona)
Joan Solà-Morales (U. Politècnica de Catalunya)

2nd meeting IST-IME
Lisboa, September 2009

Outline

1. Motivation of the problem
2. Selfadjointness with a special inner product
3. Decay of regular solutions
4. Optimality of the previous decay

Published in JDE 246 (2009), 2813-2828.

1. MOTIVATION

The problem

Wave equation with **strong damping** ($\alpha > 0$):

$$\begin{cases} u_{tt} - \alpha \triangle u_t - \triangle u = 0, \quad \Omega \times (0, \infty) \\ \\ u = 0, \quad \partial\Omega \times (0, \infty) \end{cases}$$

$\Rightarrow$ How do solutions behave when α is large?

(**overdamping** regime)

The analysis of the spectrum shows two interesting phenomena:

- $\sigma_p = \{\mu_n^{\pm}\}$ with $\mu_n^+ \to -\frac{1}{\alpha}$, $\sigma_{ess} = \{-\frac{1}{\alpha}\}$ and $\mu_n^- \to -\infty$.

  So, solutions **decay** as

$$e^{-\frac{1}{\alpha}t}$$

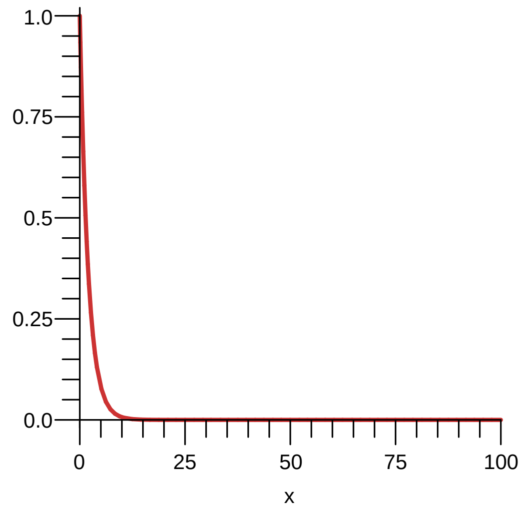
The analysis of the spectrum shows two interesting phenomena:

- $\sigma_p = \{\mu_n^{\pm}\}$ with $\mu_n^+ \to -\frac{1}{\alpha}$, $\sigma_{ess} = \{-\frac{1}{\alpha}\}$ and $\mu_n^- \to -\infty$.

  So, solutions **decay** as

$$e^{-\frac{1}{\alpha}t}$$

  Therefore, there is an **overdamping** phenomenon when α becomes large.



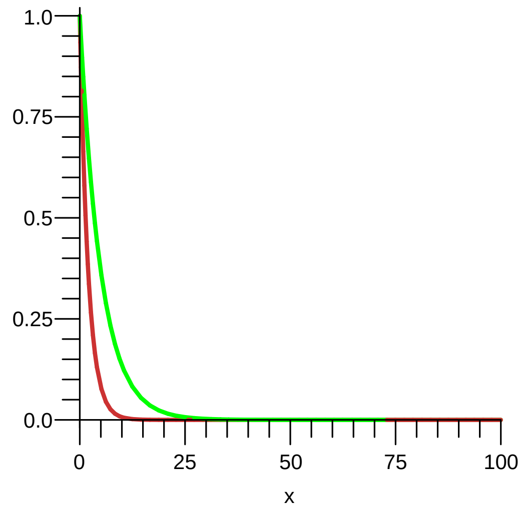
The analysis of the spectrum shows two interesting phenomena:

- $\sigma_p = \{\mu_n^{\pm}\}$ with $\mu_n^+ \to -\frac{1}{\alpha}$, $\sigma_{ess} = \{-\frac{1}{\alpha}\}$ and $\mu_n^- \to -\infty$.

  So, solutions **decay** as

$$e^{-\frac{1}{\alpha}t}$$

  Therefore, there is an **overdamping** phenomenon when α becomes large.



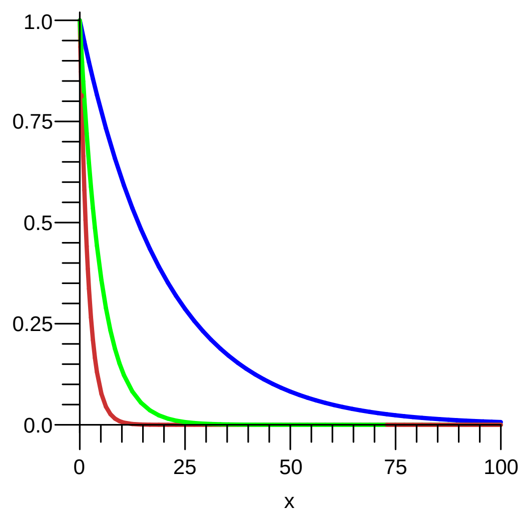
The analysis of the spectrum shows two interesting phenomena:

- $\sigma_p = \{\mu_n^{\pm}\}$ with $\mu_n^+ \to -\frac{1}{\alpha}$, $\sigma_{ess} = \{-\frac{1}{\alpha}\}$ and $\mu_n^- \to -\infty$.

  So, solutions **decay** as

$$e^{-\frac{1}{\alpha}t}$$

  Therefore, there is an **overdamping** phenomenon when α becomes large.



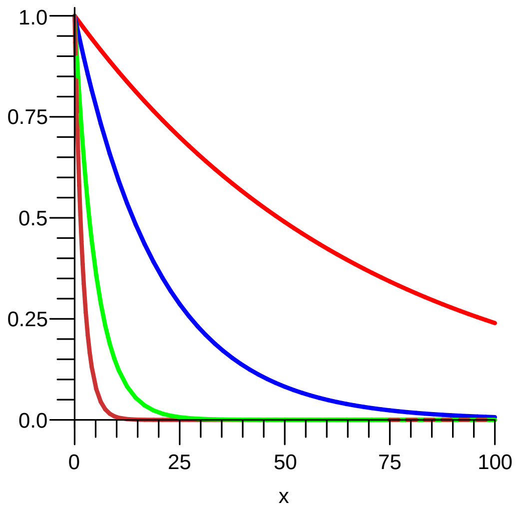
The analysis of the spectrum shows two interesting phenomena:

- $\sigma_p = \{\mu_n^{\pm}\}$ with $\mu_n^{+} \to -\frac{1}{\alpha}$, $\sigma_{ess} = \{-\frac{1}{\alpha}\}$ and $\mu_n^{-} \to -\infty$.

  So, solutions **decay** as

$$e^{-\frac{1}{\alpha}t}$$

Therefore, there is an **overdamping** phenomenon when α becomes large.



The analysis of the spectrum shows two interesting phenomena:

- $\sigma_p = \{\mu_n^{\pm}\}$ with $\mu_n^+ \to -\frac{1}{\alpha}$, $\sigma_{ess} = \{-\frac{1}{\alpha}\}$ and $\mu_n^- \to -\infty$.

  So, solutions **decay** as

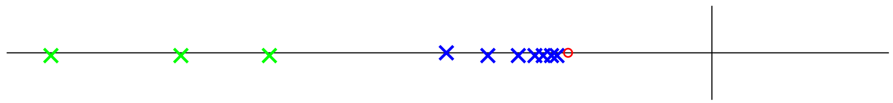
$$e^{-\frac{1}{\alpha}t}$$

  Therefore, there is an **overdamping** phenomenon when α becomes large.

QUESTION: Can we achieve a **better decay**, maybe for smoother solutions?

$$\left(\text{for instance } \frac{e^{-\frac{1}{\alpha}t}}{\sqrt{t}}, \ldots \right)$$

- $\sigma \subset \mathbb{R}$ when α is large.



QUESTION : Is there any inner product that makes the problem **selfadjoint**?

OBJECTIVE: Answer this two questions for the equation

$$u_{tt} - \alpha B u_t - A u = 0, \quad \Omega \times (0, \infty)$$

where A, B are operators satisfying certain conditions.

Answer: YES, when α is sufficiently large (overdamping regime)

Before going on: overdamping in ODEs

Let's consider

$$x'' + \alpha x' + kx = 0 \;\Leftrightarrow\; \frac{d}{dt}\begin{pmatrix} x \\ x' \end{pmatrix} = L\begin{pmatrix} x \\ x' \end{pmatrix},\; L = \begin{pmatrix} 0 & 1 \\ -k & -\alpha \end{pmatrix}$$

We want an inner product that makes L symmetric.

- We have found a product where this is possible:

$$\left(\overrightarrow{X_1}, \overrightarrow{X_2}\right)_E := \left(\begin{pmatrix} \frac{\alpha^2}{2} - k & \frac{\alpha}{2} \\ \frac{\alpha}{2} & 1 \end{pmatrix}\overrightarrow{X_1}, \overrightarrow{X_2}\right)$$

if $\boxed{\alpha > 2\sqrt{k}}$ (**overdamping** for ODEs)

- This will inspire the inner product for the PDE problem

Back to the PDE

We can write the equation as:

$$\frac{d}{dt}\begin{pmatrix} u \\ u_t \end{pmatrix} = L \begin{pmatrix} u \\ u_t \end{pmatrix}$$

with

$$L : \mathcal{D}(L) = \mathcal{D}(A) \times \mathcal{D}(B) \subset X \to X = \mathcal{D}(B) \times \mathcal{H}$$

defined as $L = \begin{pmatrix} 0 & Id \\ -A & -\alpha B \end{pmatrix}$.

Inspired in the results for the ODEs ...

2. L SELFADJOINT

Theorem 1

Suppose:

(H1) A, B strictly positive operators, selfadjoint with compact resolvent, and such that

$\mathcal{D}(A) \subset \mathcal{D}(B) \subset \mathcal{D}(A^{1/2})$ continuously.

(H2) $(Au_1, Bu_2)_{\mathcal{H}} = (Bu_1, Au_2)_{\mathcal{H}}$ for all $u_1, u_2 \in \mathcal{D}(A)$

(**commutativity**).

(H3) $\alpha > 2M$ (**overdamping** condition)

(where $\left\|A^{1/2}u\right\|_{\mathcal{H}} \leq M \left\|Bu\right\|_{\mathcal{H}}$)

Then,

- L is **selfadjoint** with the **new** inner product $(\ ,\ )_E$:

$$\left(\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right)_E := \frac{\alpha^2}{2} (\, Bu_1, Bu_2 \,)_{\mathcal{H}} - \left(A^{\frac{1}{2}} u_1, A^{\frac{1}{2}} u_2 \right)_{\mathcal{H}}$$

$$+ \frac{\alpha}{2} (\, Bu_1, v_2 \,)_{\mathcal{H}} + \frac{\alpha}{2} (\, v_1, Bu_2 \,)_{\mathcal{H}} + (\, v_1, v_2 \,)_{\mathcal{H}} \,.$$

- L is **dissipative** with this scalar product:

$$\left(L \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right)_E \leq 0, \quad \forall \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(L).$$

Some comments

- Formally, the new product is the same as the ODE one
  - ODE:

$$x'' + \alpha\, x' + k\, x = 0 \;\Rightarrow \begin{pmatrix} \frac{\alpha^2}{2} - k & \frac{\alpha}{2} \\ \frac{\alpha}{2} & 1 \end{pmatrix}$$

    when $\alpha > 2\sqrt{k}$

  - PDE:

$$u_{tt} + \alpha\, B\, u_t + Au = 0 \;\Rightarrow \begin{pmatrix} \frac{\alpha^2}{2} B^2 - A & \frac{\alpha}{2} B \\ \frac{\alpha}{2} B & \text{Id} \end{pmatrix}$$

    when $\alpha > 2M$, that is, when $\alpha \|Bu\| > 2\|A^{\frac{1}{2}} u\|$

Some comments

- Previous results:
  - first time where an inner product that makes the problem selfadjoint is given
  - real spectrum when overdamping had been observed previously [D.L.Russell'75, P.Freitas'97].

- **Corollary (known).** L generates an analytical semigroup

  (Proof. When α is large, L is selfadjoint and dissipative; for other α, perturbation of the previous case)

Some comments

- Examples
  - Case $A = B$. Commutativity condition is automatically fulfilled and overdamping condition happens when $\alpha > 2/\sqrt{\nu}$, where $(Au, u)_{\mathcal{H}} \geq \nu(u, u)_{\mathcal{H}}$, $\nu > 0$
  - Case $\mathcal{H} = L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$, $A = B = -\triangle$:

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_E^2 = \frac{\alpha^2}{2} \int_\Omega |\Delta u|^2 - \int_\Omega |\nabla u|^2 - \frac{\alpha}{2} \int_\Omega (\Delta u)\overline{v} - \frac{\alpha}{2} \int_\Omega v(\Delta \overline{u}) + \int_\Omega |v|^2$$

  - Case $A \neq B$ fulfilling the hypotheses of the theorem: $B = aA + b\, Id$, $B = A^{\frac{1}{2}}$, ... but not $B = aA + b(x)\, Id$

Proof of Theorem 1 (I)

1. $(\ ,\ )_E$ is well defined and equivalent to $(\ ,\ )_X$

We want to see that $\left\|(u,v)^T\right\|_E \geq 0$. After some calculus, we can see that we have to check if

$$\left(\frac{\alpha^2}{2} - \frac{\alpha^2}{4(1-\varepsilon)} - \varepsilon\right) \|Bu\|^2_{\mathcal{H}} \geq^? \left\|A^{\frac{1}{2}}u\right\|^2_{\mathcal{H}}$$

for some $0 < \varepsilon < 1$.

Proof of Theorem 1 (I)

1. $(\ ,\ )_E$ **is well defined and equivalent to** $(\ ,\ )_X$

We want to see that $\left\|(u,v)^T\right\|_E \geq 0$. After some calculus, we can see that we have to check if

$$\left(\frac{\alpha^2}{2} - \frac{\alpha^2}{4(1-\varepsilon)} - \varepsilon\right) \|Bu\|_{\mathcal{H}}^2 \geq^? \left\|A^{\frac{1}{2}}u\right\|_{\mathcal{H}}^2$$

for some $0 < \varepsilon < 1$. Using that

$$\lim_{\varepsilon \to 0} \left(\frac{\alpha^2}{2} - \frac{\alpha^2}{4(1-\varepsilon)} - \varepsilon\right) = \frac{\alpha^2}{4} \quad > \quad M^2$$

true if **(H3), overdamping**

Proof of Theorem 1 (I)

1. $(\ ,\ )_E$ **is well defined and equivalent to** $(\ ,\ )_X$

We want to see that $\left\|(u,v)^T\right\|_E \geq 0$. After some calculus, we can see that we have to check if

$$\left(\frac{\alpha^2}{2} - \frac{\alpha^2}{4(1-\varepsilon)} - \varepsilon\right) \|Bu\|_{\mathcal{H}}^2 \geq^? \left\|A^{\frac{1}{2}}u\right\|_{\mathcal{H}}^2$$

for some $0 < \varepsilon < 1$. Using that

$$\lim_{\varepsilon \to 0} \left(\frac{\alpha^2}{2} - \frac{\alpha^2}{4(1-\varepsilon)} - \varepsilon\right) = \frac{\alpha^2}{4} \quad > \quad M^2$$

true if **(H3), overdamping**

and also using **(H1), continuous imbedding,** we obtain that the previous inequality is fulfilled. This also allows to show the equivalence with the usual inner product $(\ ,\ )_X$. ✓

Proof (II)

2.1. L symmetric with $(\,,\,)_E$

We have to see that

$$\left(L \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right)_E \stackrel{?}{=} \left(\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, L \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right)_E$$

All the terms fulfill this because A, B are **selfadjoint (H1)** .

The only tricky term is:

$$(\,Au_1, Bu_2\,)_{\mathcal{H}} \;\stackrel{?}{=}\; (\,Bu_1, Au_2\,)_{\mathcal{H}}$$

Proof (II)

2.1. L symmetric with $(\,,\,)_E$

We have to see that

$$\left(L \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right)_E \stackrel{?}{=} \left(\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, L \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right)_E$$

All the terms fulfill this because A, B are **selfadjoint (H1)** .

The only tricky term is:

$$(\,Au_1, Bu_2\,)_{\mathcal{H}} \quad = \quad (\,Bu_1, Au_2\,)_{\mathcal{H}}$$

true if **(H2), commutativity**

Proof (II)

2.2. L invertible

We have to solve

$$\begin{pmatrix} v \\ -Au - \alpha Bv \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

which has solution as A is **strictly positive (H1)**.

Therefore, L is selfadjoint with $(\ ,\ )_E$ ✓

Proof (III)

3. L is dissipative with $(\ ,\ )_E$

We want to prove that $\left(L \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right)_E \leq 0.$

By developing it, we see that it is true if

$$\big(\text{Re}\,(\,Au, v\,)_{\mathcal{H}}\big)^2 < \frac{\alpha^2}{4}\,(\,Bv, v\,)_{\mathcal{H}} \cdot (\,Au, Bu\,)_{\mathcal{H}}$$

Proof (III)

3. L is dissipative with $(\ ,\ )_E$

We want to prove that $\left(L \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right)_E \leq 0.$

By developing it, we see that it is true if

$$\left(\text{Re}\,(Au, v)_{\mathcal{H}}\right)^2 < \frac{\alpha^2}{4}\,(Bv, v)_{\mathcal{H}} \cdot (Au, Bu)_{\mathcal{H}}$$

To see this we need **(H3)** (overdamping) and two lemmas

- **Lemma 1** $\left(A^{\frac{1}{2}}u, u\right)_{\mathcal{H}} \leq M\,(Bu, u)_{\mathcal{H}}$.
- **Lemma 2** $\left(A^{\frac{1}{2}}u_1, Bu_2\right)_{\mathcal{H}} = \left(Bu_1, A^{\frac{1}{2}}u_2\right)_{\mathcal{H}}$.

True if **common basis of eigenfunctions for A and B**

(and under the hypothesis of the Theorem) ✓.

3. DECAY OF

REGULAR SOLUTIONS

(CASE $A = B$)

Theorem 2

Consider $u_{tt} + \alpha A u_t + Au = 0$ with the same hypotheses of Theorem 1.

Theorem 2

Consider $u_{tt} + \alpha A u_t + Au = 0$ with the same hypotheses of Theorem 1.

Then:

- in general, $\left\| e^{Lt} \right\|_{\mathcal{L}(X,X)} \le e^{-\frac{1}{\alpha}t}$
- but, for each $\gamma \ge 0$, if $(u(0), v(0)) \in R_\gamma$ defined as

$$R_\gamma = \left\{ (u,v) \in \mathcal{D}(A) \times \mathcal{H}, u \in \mathcal{D}(A^{\gamma/2}), \alpha A u + v \in \mathcal{D}(A^{\gamma/2}) \right\}$$

then

$$\left\| \overrightarrow{U}(t) \right\|_E \le K \cdot \left\| \overrightarrow{U}(0) \right\|_{R_\gamma} \cdot \frac{e^{-\frac{1}{\alpha}t}}{t^{\frac{\gamma}{2}}}$$

(**exponential-polynomial decay**
for more regular solutions)

Some comments

- **Overdamping** phenomenon if only exponential decay

Some comments

- **Overdamping** phenomenon if only exponential decay
- Decay: interesting from the point of view of **control**
  ($\simeq$ rate of controllability / stabilization)

Some comments

- **Overdamping** phenomenon if only exponential decay
- Decay: interesting from the point of view of **control**
  ($\simeq$ rate of controllability / stabilization)
- Previous results
  - **Polynomial** decay for regular solutions,
    but not exponential-polynomial
  - Different **methods**:

    observability inequalities

    (Ex. Lebeau & Zuazua'99, Muñoz-Rivera & Racke'01,

    Rao & Wehbe'05, ...)

Some comments

- **Overdamping** phenomenon if only exponential decay
- Decay: interesting from the point of view of **control**
  ($\simeq$ rate of controllability / stabilization)
- Previous results
  - **Polynomial** decay for regular solutions,
    but not exponential-polynomial
  - Different **methods**:

    analysis of the spectrum and resolvent bounds

    (Ex. Liu & Rao'05, Bátkai et al.'06, Muñoz-Rivera & Quintanilla'08, ...)

Some comments

- **Overdamping** phenomenon if only exponential decay
- Decay: interesting from the point of view of **control**
  ($\simeq$ rate of controllability / stabilization)
- Previous results
  - **Polynomial** decay for regular solutions,
    but not exponential-polynomial
  - Different **methods**:

    Riesz basis (Ex. Littman & Liu'99, Zhang & Zuazua'03, ...)

    ...

    (ours: similar property using that L is selfadjoint)

Proof of Theorem 2 (I)

Characterization of eigenvalues and eigenfunctions of L

We have $\sigma(A) = \{\lambda_n\}$, eigenvalues with finite algebraic multiplicity and $\lambda_n \to +\infty$, with eigenfunctions $\{\varphi_n\}$ complete in $\mathcal{H}$. Then,

- $\sigma(L) \subset (-\infty, -1/\alpha]$ and we have two sequences of eigenvalues of L:

$$\mu_n^+ = \frac{-\lambda_n\alpha + \sqrt{\lambda_n^2\alpha^2 - 4\lambda_n}}{2} \to -\frac{1}{\alpha} \notin \sigma_p(L)$$

$$\mu_n^- = \frac{-\lambda_n\alpha - \sqrt{\lambda_n^2\alpha^2 - 4\lambda_n}}{2} \to -\infty$$

- The eigenfunctions of L, $\begin{pmatrix} \varphi_n \\ \mu_n^\pm \varphi_n \end{pmatrix}$, are a complete set in X.

(Rmk. When $A \neq B$ this characterization is more complicated.)

Proof (II)

As L is **selfadjoint**, we have a basis $\{\overrightarrow{\mathrm{e}_n^{\pm}}\}$ of orthonormal eigenfunctions in $(\ ,\ )_E$. So, if we start with the initial condition

$$\overrightarrow{U}(0) = \sum_{n=1}^{\infty} a_n \overrightarrow{\mathrm{e}_n^{+}} + \sum_{n=1}^{\infty} b_n \overrightarrow{\mathrm{e}_n^{-}}$$

then the corresponding solution is given by

$$\overrightarrow{U}(t) = \sum_{n=1}^{\infty} a_n e^{\mu_n^{+} t}\, \overrightarrow{\mathrm{e}_n^{+}} + \sum_{n=1}^{\infty} b_n e^{\mu_n^{-} t}\, \overrightarrow{\mathrm{e}_n^{-}}.$$

Therefore,

$$\left\| \overrightarrow{U}(t) \right\|_E^2 = \sum_{n=1}^{\infty} |a_n|^2 e^{2\mu_n^{+} t} + \sum_{n=1}^{\infty} |b_n|^2 e^{2\mu_n^{-} t}$$

We play with the terms and obtain

$$\left\| \overrightarrow{U}(t) \right\|_E^2 <$$

$$\left(\frac{\gamma^\gamma}{e^\gamma}\right) \cdot \left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{\left[-2\left(\mu_n^+ + \frac{1}{\alpha}\right)\right]^\gamma} + \sum_{n=1}^{\infty} \frac{|b_n|^2}{\left[-2\left(\mu_n^- + \frac{1}{\alpha}\right)\right]^\gamma} \right) \cdot \frac{e^{-\frac{2}{\alpha}t}}{t^\gamma}.$$

This will converge if

$$\boxed{\sum_{n=1}^{\infty} \frac{|a_n|^2}{\left[-2\left(\mu_n^+ + \frac{1}{\alpha}\right)\right]^\gamma} < \infty.}$$

Intuitively, this is true if

$$P^+(c.i) \in \mathcal{D}(\left(-L - \frac{1}{\alpha} Id\right)^{-\frac{\gamma}{2}})$$

How to prove it?

Developing all in series in terms of λ_n, we then have to study the convergence of:

$$\sum_{n=1}^{\infty} \left(|a_n| \lambda_n^{\frac{\gamma}{2}} \right)^2 < \infty$$

We have written it when the initial condition is given in the basis of orthonormal eigenfunctions: .

$$\overrightarrow{\mathrm{e}_n^{\pm}} = \frac{\sqrt{2}}{\sqrt{(\lambda_n^2 \alpha^2 - 4\lambda_n)}} \begin{pmatrix} \varphi_n \\ \mu_n^{\pm} \varphi_n \end{pmatrix}$$

But it will be better if we use it in another basis:

$$\begin{pmatrix} \varphi_n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi_n \end{pmatrix}.$$

So we write a_n in terms of the coeficients of the new basis, α_n, β_n:

$$a_n = \frac{-\alpha_n \mu_n^- + \beta_n}{\sqrt{2}}.$$

After this change, we can see that it is enough to see when

$$\sum_{n=1}^{\infty} \left(|\alpha_n| \lambda_n^{\frac{\gamma}{2}} \right)^2 + \left(|\alpha \lambda_n \alpha_n + \beta_n| \lambda_n^{\frac{\gamma}{2}} \right)^2 < \infty.$$

We have the convergence of the initial condition fulfills that

$$u(0) \in \mathcal{D}(A^{\gamma/2}), \; \alpha A u(0) + v(0) \in \mathcal{D}(A^{\gamma/2})$$

that is if

$$(u(0), v(0))^T \in R_\gamma$$ ✓

(Rmk. "Equivalent" condition to the intuitive one)

4. OPTIMALITY

OF THE PREVIOUS

DECAY

Theorem 3

Under the previous hypotheses and if the initial condition is in R_γ, the decay is **optimal** in the following sense: **do not exist**

$$G : R_\gamma \longrightarrow [0, \infty)$$

$$\phi : [0, \infty) \longrightarrow [0, \infty) \text{ with } \phi(t) \to 0 \text{ when } t \to \infty$$

such that

$$\left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \right\|_X \leq G\left(\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \right) \cdot \frac{e^{-\frac{1}{\alpha}t}}{t^{\frac{\gamma}{2}}} \cdot \phi(t)$$

for all $t \geq 0$ and $\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \in R_\gamma$.

Some comments

- Optimality in R_γ,
  but maybe better decay for certain elements of R_γ.

- Previous results: not many for this type of problems
  (and not for exponential-polynomial decay).

  (From the previous ones, only in Zhang & Zuazua'03;
  in Liu & Rao'05 they comment that the decay is not optimal)

Proof of Theorem 3

Suppose that it is not optimal, that is, that we have G, ϕ such that

$$\left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \right\|_X \leq G\left(\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \right) \frac{e^{-\frac{1}{\alpha}t}}{t^{\frac{\gamma}{2}}} \phi(t), \quad \forall \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \in R_\gamma$$

Then the operator $\frac{t^{\frac{\gamma}{2}} e^{\frac{1}{\alpha}t}}{\phi(t)} e^{Lt} \in \mathcal{L}(R_\gamma, X)$ is **unif. bounded**

(Banach-Steinhaus Thm.):

$$(\clubsuit) \quad t^{\frac{\gamma}{2}} e^{\frac{1}{\alpha}t} \left\| \overrightarrow{U}(t) \right\|_X \leq K \left\| \overrightarrow{U}(0) \right\|_{R_\gamma} \phi(t)$$

for all $\overrightarrow{U}(0) \in R_\gamma$, $t \geq 0$.

Proof of Theorem 3

Suppose that it is not optimal, that is, that we have G, ϕ such that

$$\left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \right\|_X \leq G\left(\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \right) \frac{e^{-\frac{1}{\alpha}t}}{t^{\frac{\gamma}{2}}} \phi(t), \quad \forall \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \in R_\gamma$$

Then the operator $\dfrac{t^{\frac{\gamma}{2}} e^{\frac{1}{\alpha}t}}{\phi(t)} e^{Lt} \in \mathcal{L}(R_\gamma, X)$ is **unif. bounded** (Banach-Steinhaus Thm.):

$$(\clubsuit) \quad t^{\frac{\gamma}{2}} e^{\frac{1}{\alpha}t} \left\| \overrightarrow{U}(t) \right\|_X \leq K \left\| \overrightarrow{U}(0) \right\|_{R_\gamma} \phi(t)$$

for all $\overrightarrow{U}(0) \in R_\gamma$, $t \geq 0$.

$\Rightarrow$ we will find i.c. and $t_n \to \infty$ that will **contradict** this.

- **Family of initial conditions s.t. $\|\overrightarrow{r_n}(0)\|_{R_\gamma} = 1$.**

$$\overrightarrow{r_n}(0) = \frac{1}{(\lambda_n + \mu_n^+) + \lambda_n^{\frac{\gamma}{2}}(\alpha\lambda_n + \mu_n^+)} \begin{pmatrix} \varphi_n \\ \mu_n^+ \varphi_n \end{pmatrix}$$

- **Family of initial conditions s.t.** $\|\overrightarrow{r_n}(0)\|_{R_\gamma} = 1$**.**

$$\overrightarrow{r_n}(0) = \frac{1}{(\lambda_n + \mu_n^+) + \lambda_n^{\frac{\gamma}{2}}(\alpha\lambda_n + \mu_n^+)} \begin{pmatrix} \varphi_n \\ \mu_n^+\varphi_n \end{pmatrix}$$

- **Sequence of** $t_n \to \infty$

$$t_n = \frac{-\gamma}{2(\mu_n^+ + \frac{1}{\alpha})}$$

We write (♣) for these initial conditions and times:

$$t^{\frac{\gamma}{2}} e^{\frac{1}{\alpha} t} \left\| \overrightarrow{U}(t) \right\|_X \leq K \left\| \overrightarrow{U}(0) \right\|_{R_\gamma} \phi(t)$$

$$\Downarrow$$

$$t_n^{\frac{\gamma}{2}} e^{\frac{1}{\alpha} t_n} \|\overrightarrow{r_n}(t)\|_X \leq K \cdot 1 \cdot \phi(t_n)$$

$$\Downarrow$$

$$t_n^{\frac{\gamma}{2}} \frac{\lambda_n + |\mu_n^+|}{(\lambda_n + |\mu_n^+|) + \lambda_n^{\frac{\gamma}{2}} (1 + |\alpha \lambda_n + \mu_n^+|)} e^{-\frac{\gamma}{2}} \leq K \cdot \phi(t_n)$$

It is clear that $\lim_{n \to \infty} (K \phi(t_n)) = 0$. And the left hand side?

Developing in series in terms of $\lambda_n \to \infty$:

$$\text{left} \simeq \frac{\left(\frac{\gamma\alpha^3}{2}\right)^{\frac{\gamma}{2}}}{\alpha} + O\left(\frac{1}{\lambda_n}\right) \longrightarrow_{n\to\infty} \frac{\left(\frac{\gamma\alpha^3}{2}\right)^{\frac{\gamma}{2}}}{\alpha}$$

So, when $n \to \infty$ we can say that we can deduce from (♣) that

$$\boxed{\frac{\left(\frac{\gamma\alpha^3}{2}\right)^{\frac{\gamma}{2}}}{\alpha} \leq 0, \ \alpha > 0}$$

CONTRADICTION !!!

Developing in series in terms of $\lambda_n \to \infty$:

$$\text{left} \simeq \frac{\left(\frac{\gamma\alpha^3}{2}\right)^{\frac{\gamma}{2}}}{\alpha} + O\left(\frac{1}{\lambda_n}\right) \xrightarrow[n\to\infty]{} \frac{\left(\frac{\gamma\alpha^3}{2}\right)^{\frac{\gamma}{2}}}{\alpha}$$

So, when $n \to \infty$ we can say that we can deduce from (♣) that

$$\boxed{\frac{\left(\frac{\gamma\alpha^3}{2}\right)^{\frac{\gamma}{2}}}{\alpha} \leq 0, \;\; \alpha > 0}$$

CONTRADICTION !!!

Therefore, it has been proved the **optimality of the decay** en R_γ ✓.