Selfadjointness
and
optimal decay rates
in overdamped equations

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Outline

1. Motivation of the problem
2. Selfadjointness with a special inner product
3. Decay of regular solutions
4. Optimality of the previous decay

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1. MOTIVATION
The problem

Wave equation with strong damping ($\alpha > 0$):

$$
\begin{cases}
  u_{tt} - \alpha \Delta u_t - \Delta u = 0, & \Omega \times (0, \infty) \\
  u = 0, & \partial \Omega \times (0, \infty)
\end{cases}
$$

⇒ How do solutions behave when $\alpha$ is large? (overdamping regime)
The analysis of the spectrum shows two interesting phenomena:

\[ \sigma_p = \{ \mu_n^{\pm} \} \quad \text{with} \quad \mu_n^+ \to -\frac{1}{\alpha}, \quad \sigma_{\text{ess}} = \{-\frac{1}{\alpha}\} \quad \text{and} \quad \mu_n^- \to -\infty. \]

So, solutions decay as

\[ e^{-\frac{1}{\alpha} t} \]
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So, solutions decay as

$$e^{-\frac{1}{\alpha}t}$$

Therefore, there is an **overdamping** phenomenon when $\alpha$ becomes large.

**QUESTION**: Can we achieve a better decay, maybe for smoother solutions?

$$\left(\text{for instance } \frac{e^{-\frac{1}{\alpha}t}}{\sqrt{t}}, \ldots\right)$$
$\sigma \subset \mathbb{R}$ when $\alpha$ is large.

**QUESTION**: Is there any inner product that makes the problem *selfadjoint*?
OBJECTIVE: Answer this two questions for the equation

\[ u_{tt} - \alpha Bu_t - Au = 0, \quad \Omega \times (0, \infty) \]

where \( A, B \) are operators satisfying certain conditions.

**Answer:** YES, when \( \alpha \) is sufficiently large (overdamping regime)
Before going on: overdamping in ODEs

Let’s consider

\[ x'' + \alpha x' + kx = 0 \iff \frac{d}{dt} \begin{pmatrix} x \\ x' \end{pmatrix} = L \begin{pmatrix} x \\ x' \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 \\ -k & -\alpha \end{pmatrix} \]

We want an inner product that makes \( L \) symmetric.

- We have found a product where this is possible:

\[
\left( \vec{X}_1, \vec{X}_2 \right)_E := \left( \begin{pmatrix} \alpha^2/2 & -k & \alpha/2 \\ \alpha/2 & 1 \end{pmatrix} \right) \vec{X}_1, \vec{X}_2
\]

if \( \alpha > 2\sqrt{k} \) (overdamping for ODEs)

- This will inspire the inner product for the PDE problem
Back to the PDE

We can write the equation as:

$$\frac{d}{dt} \begin{pmatrix} u \\ u_t \end{pmatrix} = L \begin{pmatrix} u \\ u_t \end{pmatrix}$$

with

$$L : \mathcal{D}(L) = \mathcal{D}(A) \times \mathcal{D}(B) \subset X \to X = \mathcal{D}(B) \times \mathcal{H}$$

defined as

$$L = \begin{pmatrix} 0 & Id \\ -A & -\alpha B \end{pmatrix}.$$

Inspired in the results for the ODEs ...
2. L SELFADJOINT
Theorem 1

Suppose:

(H1) $A, B$ strictly positive operators, selfadjoint with compact resolvent, and such that

\[ \mathcal{D}(A) \subset \mathcal{D}(B) \subset \mathcal{D}(A^{1/2}) \] continuously.

(H2) $(Au_1, Bu_2)_\mathcal{H} = (Bu_1, Au_2)_\mathcal{H}$ for all $u_1, u_2 \in \mathcal{D}(A)$ (commutativity).

(H3) $\alpha > 2M$ (overdamping condition)

(where $\|A^{1/2}u\|_\mathcal{H} \leq M \|Bu\|_\mathcal{H}$)
Then,

$L$ is **selfadjoint** with the **new** inner product $( , )_E :$

$$
\begin{pmatrix}
\begin{pmatrix}
u_1 \\
v_2
\end{pmatrix}, \\
\begin{pmatrix}
u_2 \\
v_2
\end{pmatrix}
\end{pmatrix}_E := \frac{\alpha^2}{2} (Bu_1, Bu_2)_H - (A^{\frac{1}{2}} u_1, A^{\frac{1}{2}} u_2)_H \\
+ \frac{\alpha}{2} (Bu_1, v_2)_H + \frac{\alpha}{2} (v_1, Bu_2)_H + (v_1, v_2)_H.
$$

$L$ is **dissipative** with this scalar product:

$$
\begin{pmatrix}
L \\
v
\end{pmatrix}, \\
\begin{pmatrix}
u \\
v
\end{pmatrix}
\end{pmatrix}_E \leq 0, \quad \forall \begin{pmatrix}
u \\
v
\end{pmatrix} \in \mathcal{D}(L).
$$
Some comments

Formally, the new product is the same as the ODE one

ODE:

\[ x'' + \alpha x' + k x = 0 \Rightarrow \left( \begin{array}{cc} \frac{\alpha^2}{2} - k & \frac{\alpha}{2} \\ \frac{\alpha}{2} & 1 \end{array} \right) \]

when \( \alpha > 2\sqrt{k} \)

PDE:

\[ u_{tt} + \alpha B u_t + Au = 0 \Rightarrow \left( \begin{array}{cc} \frac{\alpha^2 B^2}{2} - A & \frac{\alpha}{2} B \\ \frac{\alpha}{2} B & \text{Id} \end{array} \right) \]

when \( \alpha > 2M \), that is, when \( \alpha \|Bu\| > 2\|A^{\frac{1}{2}}u\| \)
Some comments

- Previous results:
  - first time where an inner product that makes the problem selfadjoint is given
  - real spectrum when overdamping had been observed previously [D.L. Russell’75, P. Freitas’97].

- Corollary (known). $L$ generates an analytical semigroup

  (Proof. When $\alpha$ is large, $L$ is selfadjoint and dissipative; for other $\alpha$, perturbation of the previous case)
Some comments

Examples

- Case $A = B$. Commutativity condition is automatically fulfilled and overdamping condition happens when $\alpha > 2 / \sqrt{\nu}$, where $(Au, u)_\mathcal{H} \geq \nu (u, u)_\mathcal{H}$, $\nu > 0$

- Case $\mathcal{H} = L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$, $A = B = -\triangle$:

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_E^2 = \frac{\alpha^2}{2} \int_\Omega |\Delta u|^2 - \int_\Omega |\nabla u|^2$$

$$- \frac{\alpha}{2} \int_\Omega (\Delta u) \overline{v} - \frac{\alpha}{2} \int_\Omega v (\Delta \overline{u}) + \int_\Omega |v|^2$$

- Case $A \neq B$ fulfilling the hypotheses of the theorem:

$B = aA + b Id$, $B = A^{1/2}$, ... but not $B = aA + b(x) Id$
Proof of Theorem 1 (I)

1. $(\cdot, \cdot)_E$ is well defined and equivalent to $(\cdot, \cdot)_X$

We want to see that $\|(u, v)^T\|_E \geq 0$. After some calculus, we can see that we have to check if

$$\left(\frac{\alpha^2}{2} - \frac{\alpha^2}{4(1 - \varepsilon)} - \varepsilon\right) \|B u\|_\mathcal{H}^2 \geq \|A^{\frac{1}{2}} u\|_\mathcal{H}^2$$

for some $0 < \varepsilon < 1$. 
Proof of Theorem 1 (I)

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\]

for some \(0 < \varepsilon < 1\). Using that 

\[
\lim_{\varepsilon \to 0} \left( \frac{\alpha^2}{2} - \frac{\alpha^2}{4(1 - \varepsilon)} - \varepsilon \right) = \frac{\alpha^2}{4} > M^2
\]

true if (H3), overdamping
Proof of Theorem 1 (I)

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\]

true if (H3), overdamping

and also using (H1), continuous imbedding, we obtain that the previous inequality is fulfilled. This also allows to show the equivalence with the usual inner product \((, , X\). √
Proof (II)

2.1. $L$ symmetric with $(\cdot, \cdot)_E$

We have to see that

$$
\left( L \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right)_E = \left( \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, L \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right)_E
$$

All the terms fulfill this because $A, B$ are selfadjoint (H1).

The only tricky term is:

$$
( Au_1, Bu_2 )_H = ? \ ( Bu_1, Au_2 )_H
$$
Proof (II)

2.1. \( L \) symmetric with \((\cdot, \cdot)_E\)

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All the terms fulfill this because \( A, B \) are selfadjoint (H1).

The only tricky term is:

\[
( Au_1, Bu_2 )_\mathcal{H} = ( Bu_1, Au_2 )_\mathcal{H}
\]

true if (H2), commutativity
Proof (II)

2.2. $L$ invertible

We have to solve

$$\begin{pmatrix}
 v \\
 -Au - \alpha Bv
\end{pmatrix} = \begin{pmatrix}
 f \\
 g
\end{pmatrix}$$

which has solution as $A$ is strictly positive (H1).

Therefore, $L$ is selfadjoint with $(\cdot, \cdot)_E$. 

✓
Proof (III)

3. \( L \) is dissipative with \( (\cdot, \cdot)_E \)

We want to prove that

\[
\left( L \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right)_E \leq 0.
\]

By developing it, we see that it is true if

\[
(\text{Re} (Au, v)_\mathcal{H})^2 < \frac{\alpha^2}{4} (Bv, v)_\mathcal{H} \cdot (Au, Bu)_\mathcal{H}
\]
Proof (III)

3. \( L \) is dissipative with \((\cdot, \cdot)_E\)

We want to prove that \( L \left( \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right)_E \leq 0 \).

By developing it, we see that it is true if

\[
\left( \text{Re} \left( A u, v \right)_H \right)^2 < \frac{\alpha^2}{4} \left( B v, v \right)_H \cdot \left( A u, B u \right)_H
\]

To see this we need \((H3)\) (overdamping) and two lemmas

- **Lemma 1** \( \left( A^{1/2} u, u \right)_H \leq M \left( B u, u \right)_H \).

- **Lemma 2** \( \left( A^{1/2} u_1, B u_2 \right)_H = \left( B u_1, A^{1/2} u_2 \right)_H \).

True if **common basis of eigenfunctions for** \( A \) **and** \( B \)

(and under the hypothesis of the Theorem) \( \checkmark \).
3. DECAY OF
REGULAR SOLUTIONS
(CASE $A = B$)
Theorem 2

Consider \( u_{tt} + \alpha A u_t + Au = 0 \) with the same hypotheses of Theorem 1.
Theorem 2

Consider \( u_{tt} + \alpha Au_t + Au = 0 \) with the same hypotheses of Theorem 1.

Then:
- in general, \( \| e^{Lt} \|_{\mathcal{L}(X,X)} \leq e^{-\frac{1}{\alpha}t} \)
- but, for each \( \gamma \geq 0 \), if \( (u(0), v(0)) \in R_\gamma \) defined as

\[
R_\gamma = \{ (u, v) \in \mathcal{D}(A) \times \mathcal{H}, u \in \mathcal{D}(A^{\gamma/2}), \alpha Au + v \in \mathcal{D}(A^{\gamma/2}) \}
\]

then

\[
\| \overrightarrow{U}(t) \|_E \leq K \cdot \| \overrightarrow{U}(0) \|_{R_\gamma} \cdot \frac{e^{-\frac{1}{\alpha}t}}{t^{\gamma/2}}
\]

(exponential-polynomial decay for more regular solutions)
Some comments

- **Overdamping** phenomenon if only exponential decay
Some comments

- Overdamping phenomenon if only exponential decay
- Decay: interesting from the point of view of control
  (\(\approx\) rate of controllability / stabilization)
Some comments

- **Overdamping** phenomenon if only exponential decay

- Decay: interesting from the point of view of *control*  
  \( \approx \) rate of controllability / stabilization

- Previous results
  - **Polynomial** decay for regular solutions, but not exponential-polynomial

- Different **methods**:
  
  observability inequalities  
  (Ex. Lebeau & Zuazua’99, Muñoz-Rivera & Racke’01, Rao & Wehbe’05, ...)

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- p.21
Some comments

- **Overdamping** phenomenon if only exponential decay

- Decay: interesting from the point of view of **control**
  
  ( ≃ rate of controllability / stabilization )

- Previous results

- **Polynomial** decay for regular solutions, but not exponential-polynomial

- Different **methods**:

  analysis of the spectrum and resolvent bounds

  (Ex. Liu & Rao’05, Bátkai et al.’06, Muñoz-Rivera & Quintanilla’08, ...)

- p.21
Some comments

- **Overdamping** phenomenon if only exponential decay

- Decay: interesting from the point of view of **control**
  (\(\sim\) rate of controllability / stabilization)

- Previous results
  - **Polynomial** decay for regular solutions, but not exponential-polynomial

- Different **methods**:

  Riesz basis (Ex. Littman & Liu’99, Zhang & Zuazua’03, ...)

  ...

  (ours: similar property using that \(L\) is selfadjoint)
Proof of Theorem 2 (I)

Characterization of eigenvalues and eigenfunctions of $L$

We have $\sigma(A) = \{\lambda_n\}$, eigenvalues with finite algebraic multiplicity and $\lambda_n \to +\infty$, with eigenfunctions $\{\varphi_n\}$ complete in $\mathcal{H}$. Then, $\sigma(L) \subset (-\infty, -1/\alpha]$ and we have two sequences of eigenvalues of $L$:

$$\mu^+_n = \frac{-\lambda_n \alpha + \sqrt{\lambda_n^2 \alpha^2 - 4\lambda_n}}{2} \to -\frac{1}{\alpha} \notin \sigma_p(L)$$

$$\mu^-_n = \frac{-\lambda_n \alpha - \sqrt{\lambda_n^2 \alpha^2 - 4\lambda_n}}{2} \to -\infty$$

The eigenfunctions of $L$, $\begin{pmatrix} \varphi_n \\ \mu^+_n \varphi_n \end{pmatrix}$, are a complete set in $X$.

(Rmk. When $A \neq B$ this characterization is more complicated.)
Proof (II)

As $L$ is selfadjoint, we have a basis $\{e_n^\pm\}$ of orthonormal eigenfunctions in $(\cdot, \cdot)_E$. So, if we start with the initial condition

$$\vec{U}(0) = \sum_{n=1}^{\infty} a_n e_n^+ + \sum_{n=1}^{\infty} b_n e_n^-$$

then the corresponding solution is given by

$$\vec{U}(t) = \sum_{n=1}^{\infty} a_n e^{\mu_n^+ t} e_n^+ + \sum_{n=1}^{\infty} b_n e^{\mu_n^- t} e_n^-.$$ 

Therefore,

$$\|\vec{U}(t)\|^2_E = \sum_{n=1}^{\infty} |a_n|^2 e^{2\mu_n^+ t} + \sum_{n=1}^{\infty} |b_n|^2 e^{2\mu_n^- t}.$$
We play with the terms and obtain

\[
\left\| \overrightarrow{U}(t) \right\|_E^2 < \\
\left( \frac{\gamma^\gamma}{e^\gamma} \right) \cdot \left( \sum_{n=1}^{\infty} \frac{|a_n|^2}{[-2(\mu_n^+ + \frac{1}{\alpha})]^{\gamma}} + \sum_{n=1}^{\infty} \frac{|b_n|^2}{[-2(\mu_n^- + \frac{1}{\alpha})]^{\gamma}} \right) \cdot e^{-\frac{2}{\alpha}t}.
\]

This will converge if

\[
\sum_{n=1}^{\infty} \frac{|a_n|^2}{[-2(\mu_n^+ + \frac{1}{\alpha})]^{\gamma}} < \infty.
\]

Intuitively, this is true if

\[
P^+(c.i) \in \mathcal{D}\left( \left( -L - \frac{1}{\alpha} \text{Id} \right)^{-\frac{\gamma}{2}} \right)
\]
How to prove it?

Developing all in series in terms of $\lambda_n$, we then have to study the convergence of:

$$\sum_{n=1}^{\infty} \left( |a_n| \lambda_n^{\frac{\gamma}{2}} \right)^2 < \infty$$
We have written it when the initial condition is given in the basis of orthonormal eigenfunctions:

\[ \vec{e}_{n}^{\pm} = \frac{\sqrt{2}}{\sqrt{\left(\lambda_{n}^{2}\alpha^{2} - 4\lambda_{n}\right)}} \begin{pmatrix} \varphi_{n} \\ \mu_{n}^{\pm} \varphi_{n} \end{pmatrix} \]

But it will be better if we use it in another basis:

\[
\begin{pmatrix} \varphi_{n} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \varphi_{n} \end{pmatrix}.
\]

So we write \( a_{n} \) in terms of the coefficients of the new basis, \( \alpha_{n}, \beta_{n} \):

\[ a_{n} = \frac{-\alpha_{n}\mu_{n}^{-} + \beta_{n}}{\sqrt{2}}. \]
After this change, we can see that it is enough to see when

\[ \sum_{n=1}^{\infty} \left( |\alpha_n| \lambda_n^{\frac{\gamma}{2}} \right)^2 + \left( |\alpha_n \alpha' + \beta_n| \lambda_n^{\frac{\gamma}{2}} \right)^2 < \infty. \]

We have the convergence of the initial condition fulfills that

\[ u(0) \in \mathcal{D}(A^{\gamma/2}), \; \alpha Au(0) + v(0) \in \mathcal{D}(A^{\gamma/2}) \]

that is if

\[ (u(0), v(0))^T \in R_\gamma \]

(Rmk. “Equivalent” condition to the intuitive one)
4. OPTIMALITY OF THE PREVIOUS DECAY
Theorem 3

Under the previous hypotheses and if the initial condition is in $R_\gamma$, the decay is **optimal** in the following sense: do not exist

$$G : R_\gamma \rightarrow [0, \infty)$$

$$\phi : [0, \infty) \rightarrow [0, \infty) \text{ with } \phi(t) \rightarrow 0 \text{ when } t \rightarrow \infty$$

such that

$$\left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \right\|_X \leq G \left( \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \right) \cdot \frac{e^{-\frac{1}{\alpha}t}}{t^{\frac{\gamma}{2}}} \cdot \phi(t)$$

for all $t \geq 0$ and $\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \in R_\gamma$. 
Some comments

- Optimality in $R_\gamma$, but maybe better decay for certain elements of $R_\gamma$.

- Previous results: not many for this type of problems (and not for exponential-polynomial decay).

  (From the previous ones, only in Zhang & Zuazua’03; in Liu & Rao’05 they comment that the decay is not optimal)
Proof of Theorem 3

Suppose that it is not optimal, that is, that we have $G, \phi$ such that

$$\left\lVert \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \right\rVert_X \leq G \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \frac{e^{-\frac{1}{\alpha}t}}{t^{\frac{\gamma}{2}}} \phi(t), \quad \forall \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \in R_\gamma$$

Then the operator $\frac{t^{\frac{\gamma}{2}} e^{\frac{1}{\alpha}t}}{\phi(t)} e^{Lt} \in \mathcal{L}(R_\gamma, X)$ is unif. bounded (Banach-Steinhaus Thm.):

$$(\clubsuit) \quad t^{\frac{\gamma}{2}} e^{\frac{1}{\alpha}t} \left\lVert \vec{U}(t) \right\rVert_X \leq K \left\lVert \vec{U}(0) \right\rVert_{R_\gamma} \phi(t)$$

for all $\vec{U}(0) \in R_\gamma$, $t \geq 0$. 
Proof of Theorem 3

Suppose that it is not optimal, that is, that we have \( G, \phi \) such that

\[
\left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \right\|_X \leq G \left( \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \right) \frac{e^{-\frac{1}{\alpha} t}}{t^{\gamma/2}} \phi(t), \quad \forall \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \in R_\gamma
\]

Then the operator \( t^{\gamma/2} e^{\frac{1}{\alpha} t} e^{Lt} \in \mathcal{L}(R_\gamma, X) \) is unif. bounded (Banach-Steinhaus Thm.):

\[
\left( \bigspadesuit \right) \quad t^{\gamma/2} e^{\frac{1}{\alpha} t} \left\| \overrightarrow{U}(t) \right\|_X \leq K \left\| \overrightarrow{U}(0) \right\|_{R_\gamma} \phi(t)
\]

for all \( \overrightarrow{U}(0) \in R_\gamma, \ t \geq 0. \)

\( \Rightarrow \) we will find i.c. and \( t_n \to \infty \) that will contradict this.
Family of initial conditions s.t. $\| \overrightarrow{r_n}(0) \|_{R_\gamma} = 1$.

$$\overrightarrow{r_n}(0) = \frac{1}{(\lambda_n + \mu_n^+) + \lambda_n^2 (\alpha \lambda_n + \mu_n^+) (\varphi_n)} \begin{pmatrix} \varphi_n \\ \mu_n^+ \varphi_n \end{pmatrix}$$
Family of initial conditions s.t. \(\|\vec{r}_n(0)\|_{R_\gamma} = 1\).

\[
\vec{r}_n(0) = \frac{1}{(\lambda_n + \mu_n^+) + \lambda_n^2 (\alpha \lambda_n + \mu_n^+)} \begin{pmatrix}
\varphi_n \\
\mu_n^+ \varphi_n
\end{pmatrix}
\]

Sequence of \(t_n \to \infty\)

\[
t_n = \frac{-\gamma}{2(\mu_n^+ + \frac{1}{\alpha})}
\]
We write (♣) for these initial conditions and times:

\[ t^{\gamma/2} e^{\frac{1}{\alpha} t} \left\| \vec{U}(t) \right\|_X \leq K \left\| \vec{U}(0) \right\|_{R_\gamma} \phi(t) \]

\[ \Downarrow \]

\[ t_n^{\gamma/2} e^{\frac{1}{\alpha} t_n} \left\| \vec{r}(t) \right\|_X \leq K \cdot 1 \cdot \phi(t_n) \]

\[ \Downarrow \]

\[ t_n^{\gamma/2} \frac{\lambda_n + |\mu_n^+|}{(\lambda_n + |\mu_n^+|) + \lambda_n^{\gamma/2} (1 + |\alpha \lambda_n + \mu_n^+|)} e^{-\gamma^2/2} \leq K \cdot \phi(t_n) \]

It is clear that \( \lim_{n \to \infty} (K \phi(t_n)) = 0 \). And the left hand side?
Developing in series in terms of $\lambda_n \to \infty$:

$$\left(\frac{\gamma \alpha^3}{2}\right)^{\frac{\gamma}{2}} \approx \frac{\left(\frac{\gamma \alpha^3}{2}\right)^{\frac{\gamma}{2}}}{\alpha} + O\left(\frac{1}{\lambda_n}\right) \quad \rightarrow_{n \to \infty} \frac{\left(\frac{\gamma \alpha^3}{2}\right)^{\frac{\gamma}{2}}}{\alpha}$$

So, when $n \to \infty$ we can say that we can deduce from (♣) that

$$\left(\frac{\gamma \alpha^3}{2}\right)^{\frac{\gamma}{2}} \leq 0, \quad \alpha > 0$$

CONTRADICTION !!!
Developing in series in terms of $\lambda_n \rightarrow \infty$:

\[
\text{left } \simeq \left( \frac{\gamma \alpha^3}{2} \right)^{\frac{\gamma}{2}} \alpha + O \left( \frac{1}{\lambda_n} \right) \xrightarrow{n \rightarrow \infty} \left( \frac{\gamma \alpha^3}{2} \right)^{\frac{\gamma}{2}} \alpha
\]

So, when $n \rightarrow \infty$ we can say that we can deduce from (♣) that

\[
\left( \frac{\gamma \alpha^3}{2} \right)^{\frac{\gamma}{2}} \frac{1}{\alpha} \leq 0, \quad \alpha > 0
\]

CONTRADICTION !!!

Therefore, it has been proved the optimality of the decay $R_\gamma$. 