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Analysis of a viscoelastic spring–mass model ☆

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Abstract

In this paper we consider a linear wave equation with strong damping and dynamical boundary conditions as an alternative model for the classical spring–mass–damper ODE. Our purpose is to compare analytically these two approaches to the same physical system. We take a functional analysis point of view based on semigroup theory, spectral perturbation analysis and dominant eigenvalues.

Keywords: Strongly damped wave equation; Dynamical boundary conditions; Asymptotic behavior; Dominant eigenvalues

1. Introduction

Consider the motion of a system consisting of a spring of recovery constant k that is fixed at one end and attached to a rigid mass m at the other one. Suppose also that the mass movement is linearly damped by a friction force of coefficient d. Typically, the dynamics of this system is modelled by the second order differential equation

$$mu''(t) = -ku(t) - du'(t), \tag{1}$$

where $u(t)$ is the position of the mass at time t. This model considers the spring–mass–damper system as a problem with only two degrees of freedom. A more detailed point

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of view would lead us to treat the spring as a continuous medium where the deformation depends on the point, taking into account possible internal deformation differences, and also to consider its internal viscosity, apart from the external damper dissipation. This gives us a partial differential equation model in which the action of the external damper onto the mass movement appears only in the boundary conditions. An example of such a system would be the car shock absorbers, where the damper acts onto the viscoelastic spring through the wheel of the car only. Our objective is to discuss this alternative partial differential equation model and analyze its solutions, comparing their asymptotic behavior with the ones of the ordinary differential equation model.

The partial differential equation model in the appropriate variables system, justified in detail in Section 2, is the following:

$$\begin{cases} u_{tt} - u_{xx} - \alpha u_{txx} = 0, \quad 0 < x < 1,\ t > 0, \\ u(0,t) = 0, \\ u_{tt}(1,t) = -\varepsilon[u_x(1,t) + \alpha u_{tx}(1,t) + r u_t(1,t)], \end{cases} \tag{2}$$

where $\alpha > 0$ is a parameter related with the spring internal dissipation coefficient, $r > 0$ comes from the damper viscosity coefficient and $\varepsilon \geqslant 0$ is a parameter depending on the mass m and on the density and the length of the spring. We denote by $u(x,t)$ the displacement at time t of the x particle of the spring. That means that $x_0 + u(x_0, t_0)$ is the position at time t_0 of the particle of position x_0 at equilibrium. Equation (2) is a wave equation with strong damping (also called Kelvin–Voigt damping) and dynamical boundary conditions.

The interaction of an elastic medium with a rigid mass has been studied in the mathematical literature by several authors, under different points of view. With the goal of studying the controllability of the system, for example, the interaction between beams and rigid masses has been considered by C.M. Castro and E. Zuazua (see, for instance, [1]). Also, models of wave equations but with weak (or Maxwell) damping, instead of the Kelvin–Voigt one, coupled by rigid masses have been considered by A. Freiria Neves, H. de Souza Ribeiro and O. Lopes (see, for example, [11]), mainly with the aim of studying the decay of the solutions.

This important problem of the decay of the solutions or asymptotic stability in elastic systems with dissipation has also been considered by other authors, such as M. Renardy in [15], for certain types of $\mathcal{C}_0$ semigroups, or K. Liu and Z. Liu in [10]. This last work is devoted to study the exponential decay of the solutions in elastic systems which model the clamped beam with localized Kelvin–Voigt damping. It is worth to say that the asymptotic stability is not the goal of our work, since we will see that our solutions are defined by an analytic semigroup, instead of one of class $\mathcal{C}^0$ as it happens in [10]. In our case the asymptotic stability of the solutions is an automatic consequence of the existence of a finite set of dominant eigenvalues.

To our knowledge, the model (2), but with $r = 0$ and $\varepsilon = \alpha = 1$ was first considered by M. Grobbelaar-van Dalsen in [7], who showed that it defines an analytic semigroup in an appropriate functional space. The same result, also with $r = 0$, for the case of a free end (that is $\varepsilon = \infty$ in (2)) was obtained in [2] with different methods. The functional framework for Eq. (2), discussed below in Section 3, is based on the approach of [7], and also on the previous work of P. Massat in [12] for the abstract equation $u_{tt} + \alpha A u_t + A_u = f(t, u, u_t)$, where A is a sectorial operator and $\alpha > 0$. This kind of equation, but with

Neumann boundary conditions, has also appeared in the work of N. Cónsul and J. Solà-Morales [3].

To compare Eq. (2) with the classical ODE (1) our main tool will be the *dominant eigenvalues*. This is a very simple and well known idea: when the solutions of a differential equation like (2) are of the form $u(x,t) = \sum a_n e^{\lambda_n t} u_n(x)$, their asymptotic behavior is dominated by the terms having the greatest $\mathrm{Re}(\lambda_n)$. This can simplify a model with infinitely many degrees of freedom to a finite dimensional one.

Of course, this situation is not completely simple when one deals with nonselfadjoint linear operators that also have essential spectrum, apart from the eigenvalues, which is our case. But still then, the same ideas can be used. Let us be more precise: consider an abstract evolution equation

$$\frac{d}{dt}x(t) = Bx(t), \tag{3}$$

where B is a linear operator in a Banach space $\mathcal{X}$ with spectrum $\sigma(B)$. Suppose that $\sigma(B)$ has k isolated eigenvalues with finite algebraic multiplicities, $\lambda_1, \dots, \lambda_k$, and that there exist $\omega_1, \omega_2 \in \mathbb{R}$ such that

$$\mathrm{Re}\,\lambda < \omega_2 < \omega_1 < \mathrm{Re}\,\lambda_i \quad \forall i = 1, \dots, k,\ \forall \lambda \in \sigma(B) \setminus \{\lambda_1, \dots, \lambda_k\}.$$

Then, we say that the operator B admits $\{\lambda_1, \dots, \lambda_k\}$ as a finite subset of *dominant eigenvalues*. In this situation we have a natural decomposition of the spectrum in $\sigma_1 = \{\lambda_1, \dots, \lambda_k\}$ and $\sigma_2 = \sigma(B) \setminus \{\lambda_1, \dots, \lambda_k\}$, a decomposition of the total space $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$, with $\dim(X_1) < \infty$, and of the operator, $B_1 = B|_{\mathcal{X}_1}$ and $B_2 = B|_{\mathcal{X}_2}$ (see, for example, [8]). Then, the following result can be easily deduced from the general theory of analytic semigroups (see [5,8,13]). We point out that this result needs not to hold for general C_0 semigroups: this is why the analyticity proved in [7] is important for the application to (2).

Theorem 1. *Let $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ and $B = (B_1, B_2)$ as above, and suppose that B is the infinitesimal generator of an analytic semigroup. Let $x(t) = x_1(t) + x_2(t)$, $x_1(t) \in \mathcal{X}_1$ and $x_2(t) \in \mathcal{X}_2$, be the solution of* (3) *with initial conditions $x(0) = x_1(0) + x_2(0)$ and suppose also that $x_1(0) \neq 0$. Then*

$$\lim_{t\to\infty} \frac{\|x(t) - x_1(t)\|}{\|x(t)\|} = 0. \tag{4}$$

Because of this result it is reasonable to say that the solutions of the finite dimensional ordinary differential equation $x_1' = B_1 x_1$ are a good approximation for the solutions of the infinite dimensional evolution equation $x' = Bx$ for large time (see [14] for details).

As we said before, Theorem 1 shows our approach to the description of the asymptotic behavior of the elastic model. Our study of the spectrum is not in order to find bounds of the form $\|e^{tB}\| \leqslant Me^{-\omega t}$ $(\omega > 0)$ as it is usual in the studies on asymptotic stability in elastic models, but to find the limit of the solutions in the sense of (4). As we will see in Section 4 that they tend to certain second order ODE solutions, their asymptotic behavior will be explicitly determined.

The main result of this paper is obtained in Section 4 in which we prove that for small values of ε there are two complex conjugate dominant eigenvalues for (2), so we show that the PDE has an ODE of the type of (1) as a limit. The dependence of these dominant eigenvalues with respect to ε is also calculated up to some reasonable approximation as well as the coefficients of the limit ODE. The basic tools in the proof of this main result are the characteristic equation for the eigenvalues (Eq. (14)), a control of the essential spectrum of the infinitesimal generator of the semigroup and the notion of generalized convergence of closed operators. For these last two tools we need a precise functional formulation of the problem and, in particular, a characterization of the domains of the operators involved. This is done in Section 3.

Section 2 is devoted to modelling. Our personal reason to include it is to show how natural is to consider (2) as the first generalization of the classical spring–mass–damper ODE model, as well as to show that through this natural generalization one obtains a strong damping in the wave equation, and not a weak damping as one could perhaps suspect.

We also want to say that our main result admits a nonlinear version, if one deals with a nonlinear perturbation of (2). In that case one could prove the existence of a globally attracting invariant manifold and a nonlinear limit ODE on it. This will be studied in a subsequent paper. It is also left for another occasion the study of the limit $\alpha \to 0$ and $\alpha = 0$ (for fixed $\varepsilon > 0$). In these cases the situation is very different: for instance, when small values of $\alpha > 0$ are considered the number of dominant eigenvalues can be up to four or more, all them with the same real part, and even there is no finite number of dominant eigenvalues when $\alpha = 0$. Some results in these directions can be seen in [14] or in forthcoming publications.

2. Modelling

The mechanical behavior of a viscoelastic spring of length L can be modelled by the well-known strongly damped wave equation, but then the action of the external damper onto the spring through the mass at the $x = L$ end is going to appear as a boundary condition. This boundary condition is slightly different from that considered in [7] in which the external damper does not appear. Let us derive it in detail from the rheological point of view.

The rheological approach consists of discretizing viscoelastic materials into different combinations of elementary units, which are springs and dashpots. As a spring models the material elastic behavior, its constitutive equation is given by Hooke's law $\sigma_e = E\varepsilon_e$, where σ_e is the elastic stress, E the Young modulus and ε_e the elastic strain. And as a dashpot models the viscosity, its constitutive equation is $\sigma_v = E_1\dot{\varepsilon}_v$, where $\sigma_v, \dot{\varepsilon}_v$ stand for viscous stress and strain rate and E_1 is the viscosity coefficient. These basic elements can be coupled either in series or in parallel (see [4] for more details). A parallel-coupled spring and dashpot system is known as the Kelvin–Voigt model, whose constitutive equation is $\sigma = E_1\dot{\varepsilon} + E\varepsilon$, where σ and ε are now the total stress and strain.

Let us think now our material as a sequence of increasingly many series-coupled Kelvin–Voigt systems (that is, a continuous Kelvin–Voigt model). And the last system is

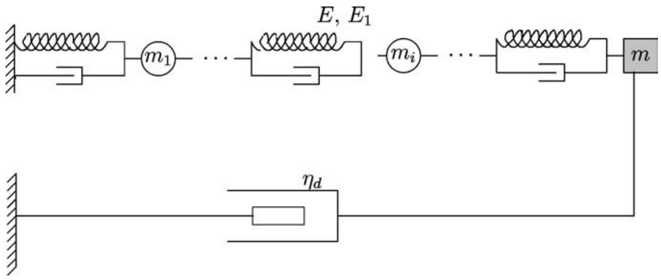

Fig. 1. Rheological model for the spring–mass–damper system.

parallel-coupled with a single external damper (see Fig. 1) across a rigid mass m. At the other end, the system is kept fixed.

Following [4], the equation of motion is given by the balance of forces between the ith and $(i+1)$th components,

$$m_i \frac{d^2 u_i(t)}{dt^2} = \sigma_{i+1} - \sigma_i, \tag{5}$$

where $u_i(t)$ is the displacement of the ith mass. Replacing the single Kelvin–Voigt equation into (5), writing $m_i = \rho_i h$ (being h the length of each component and ρ_i the local density) and writing also the strain in terms of displacement, Eq. (5) becomes

$$\rho_i \frac{d^2 u_i(t)}{dt^2} = E_1 \frac{d}{dt}\left[\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right] + E\left[\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right]. \tag{6}$$

Taking the limit as $h \to 0$ in both sides of the equation, we obtain the continuous system, whose equation is

$$\rho(x) u_{tt}(x,t) = E_1 u_{xxt}(x,t) + E u_{xx}(x,t). \tag{7}$$

Actually, in our model we will consider a constant density ρ.

Concerning the boundary conditions, since the end $x = 0$ is fixed we have

$$u(0,t) = 0. \tag{8}$$

For the boundary condition at $x = L$, only the action of the last Kelvin–Voigt component and the damper have to be considered (see again Fig. 1). As these two components are parallel-coupled, we have $\sigma = (E\varepsilon_n + E_1\dot{\varepsilon_n}) + \eta_d \dot{\varepsilon}_d$, where the n-subindex stands for the last Kelvin–Voigt component and the d-subindex means that of the damper. Following the same idea as before, this equation comes into

$$m u_{tt}|_{x=L} = -\left[E \frac{u_n - u_{n-1}}{h} + E_1 \frac{d}{dt}\left(\frac{u_n - u_{n-1}}{h}\right) + \frac{\eta_d}{L} u_t\right]\Bigg|_{x=L}. \tag{9}$$

Taking limits as $h \to 0$ we obtain the dynamical boundary condition at $x = L$,

$$m u_{tt}(L,t) = -(E u_x + q u_t + E_1 u_{xt})(L,t), \tag{10}$$

where $q = \eta_d / L$.

We now apply a change of variables to (7), (8) and (10) in order to obtain the nondimensional model (2). The change of variables is

$$x \leftrightarrow \frac{x}{L}, \quad t \leftrightarrow \frac{t\sqrt{E/\rho}}{L}.$$

So now the length of the system is 1. We also give a change of functions,

$$u \leftrightarrow \frac{u}{L}.$$

The nondimensional parameter change is

$$\alpha = \frac{E_1}{\sqrt{E\rho}L}, \quad \varepsilon = \frac{\rho L}{m}, \quad r = \frac{q}{\sqrt{E\rho}}.$$

Our model (2) now depends on the three nonnegative nondimensional parameters α, r and ε. To get some intuition about these parameters we observe that for fixed E, ρ and L, we have that α comes from the internal spring viscosity E_1, r comes from the external damper coefficient q and $1/\varepsilon$ is proportional to m, the rigid mass at the end.

3. Functional setting

In this section, let us think in the model (2) as the following Cauchy problem:

$$\begin{cases} u_{tt} - u_{xx} - \alpha u_{txx} = 0, \quad 0 < x < 1,\ t > 0, \\ u(0,t) = 0, \quad t > 0, \\ u_{tt}(1,t) + \varepsilon u_x(1,t) + \varepsilon\alpha u_{tx}(1,t) + \varepsilon r u_t(1,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x), \quad 0 < x < 1, \\ u_t(x,0) = v_0(x), \quad 0 < x < 1, \\ u(1,0) = \eta\ (= u_0(1)), \\ u_t(1,0) = \mu\ (= v_0(1)), \end{cases} \tag{11}$$

with $u = u(x,t)$, $t \in [0,\infty)$ and $\alpha > 0$, $\varepsilon, r > 0$.

We now give a functional framework that will be appropriate for obtaining existence and uniqueness of solutions for (11), and also for discussing the convergence of the spectra of some of the involved operators. This convergence is obtained by using the notion of generalized convergence of operators (see Section 4 below). This has been the motivation of our careful choice of some of the function spaces here.

We want to write (11) as an evolution equation, in a similar way as it is done by Grobbelaar in [7] or by Massat in [12]. Let us consider the following spaces:

$$X_2 = \{(u,\gamma) \in H^2(0,1) \times \mathbb{C},\ u(1) = \gamma,\ u(0) = 0\}$$

as a subspace of $H^2(0,1) \times \mathbb{C}$;

$$X_1 = \{(u,\gamma) \in H^1(0,1) \times \mathbb{C},\ u(1) = \gamma,\ u(0) = 0\}$$

as a subspace of $H^1(0,1) \times \mathbb{C}$; and

$$X_0 = \{(u,\gamma) \in L^2(0,1) \times \mathbb{C}\} = L^2(0,1) \times \mathbb{C}.$$

In these subspaces, a natural inner product is defined:

$$\big\langle\big(u, u(1)\big), \big(v, v(1)\big)\big\rangle_{X_1} = \int_0^1 u_x \overline{v_x}\, dx$$

and

$$\big\langle(u, \gamma), (v, \beta)\big\rangle_{X_0} = \int_0^1 u\bar{v}\, dx \; + \frac{1}{\varepsilon}\gamma\overline{\beta}.$$

It can be easily proved that this products are equivalent to those defined in the Sobolev space in which are included (see [14]). This ε-dependence of the inner product on X_0 will be specially useful in some of the proofs below.

We define $(A_\alpha, \mathcal{D}(A_\alpha))$ as follows. The domain $\mathcal{D}(A_\alpha)$ is

$$\mathcal{D}(A_\alpha) = \left\{ \begin{pmatrix} (u, u(1)) \\ (v, v(1)) \end{pmatrix} \in X_1 \times X_1,\ (u + \alpha v) \in H^2(0, 1) \right\} \subset \mathcal{H},$$

where $\mathcal{H} = X_1 \times X_0$ is a Hilbert space with the inner product

$$\left\langle \begin{pmatrix} (u_1, u_1(1)) \\ (u_0, \gamma_0)) \end{pmatrix}, \begin{pmatrix} (v_1, v_1(1)) \\ (v_0, \beta_0)) \end{pmatrix} \right\rangle_{\mathcal{H}} = \big\langle\big(u_1, u_1(1)\big), \big(v_1, v_1(1)\big)\big\rangle_{X_1} + \big\langle(u_0, \gamma_0), (v_0, \beta_0)\big\rangle_{X_0}.$$

If

$$V = \begin{pmatrix} (u, u(1)) \\ (v, v(1)) \end{pmatrix} \in \mathcal{D}(A_\alpha)$$

then we define the operator as

$$A_\alpha V = \begin{pmatrix} (v, v(1)) \\ ((u + \alpha v)_{xx}, -\varepsilon(u + \alpha v)_x(1) - \varepsilon r v(1)) \end{pmatrix}.$$

Then, for

$$V = \begin{pmatrix} (u, u(1)) \\ (u_t, u_t(1)) \end{pmatrix} \quad \text{and} \quad V(0) = F_0 = \begin{pmatrix} (u_0(x), \eta) \\ (v_0(x), \mu) \end{pmatrix},$$

Eq. (11) can be written as the evolution equation

$$\begin{cases} \frac{d}{dt} V = A_\alpha V, & t \in (0, \infty), \\ V(0) = F_0. \end{cases} \tag{12}$$

The existence and uniqueness of the solutions for (12), in terms of the generated semigroup, follows the proof given by Grobbelaar in [7], who actually is based on Massat's proof (see [12]). This result is summarized in the following theorem.

Theorem 2. *The operator $(A_\alpha, \mathcal{D}(A_\alpha))$ with $\alpha > 0$ is the infinitesimal generator of an analytic semigroup in $\mathcal{H}$.*

Idea of the proof. The idea is simply to decompose $-A_\alpha$ into

$$-A_\alpha V = BV + KV = \begin{pmatrix} (-v, -v(1)) \\ ((-u-\alpha v)_{xx}, \varepsilon(u+\alpha v)_x(1)) \end{pmatrix} + \begin{pmatrix} (0,0) \\ (0, \varepsilon r v(1)) \end{pmatrix}.$$

Applying a result of [12], we see that B is the infinitesimal generator of an analytic semigroup. As A_α is a bounded perturbation of B, it is also the infinitesimal generator of an analytic semigroup in the same space as B, which turns to be $\mathcal{H}$ (the details of this proof are given in [14]). □

4. The case of ε near 0

The case of a small positive ε and a fixed $\alpha > 0$ is a case of physical interest as it models a spring-mass system when the mass at the end is taken large. The linear operator is denoted now as $A_\alpha(\varepsilon)$, and for $\varepsilon \sim 0$ is going to be thought as a perturbation of the operator for the limit case $\varepsilon = 0$ (or an infinitely large mass), which is denoted as $A_\alpha(0)$. Our main result is the following

Theorem 3. *For fixed $\alpha, r > 0$ there exists a certain $\varepsilon_0 > 0$ (depending on α and r) for which $(A_\alpha(\varepsilon), \mathcal{D}(A_\alpha))$ when $\varepsilon < \varepsilon_0$ admits $\{\lambda_0^+(\varepsilon), \lambda_0^-(\varepsilon)\}$ as a subset of two simple dominant eigenvalues, where*

$$\lambda_0^+(\varepsilon) = i\sqrt{\varepsilon} - \frac{\alpha+r}{2}\varepsilon - i\frac{4+3(\alpha+r)^2}{24}(\sqrt{\varepsilon})^3 + \frac{\alpha+r}{6}\varepsilon^2 + i\frac{176+360(\alpha+r)^2-45(\alpha+r)^4}{5760}(\sqrt{\varepsilon})^5 - \left(\frac{2\alpha}{45}+\frac{r}{30}\right)\varepsilon^3 + O\big((\sqrt{\varepsilon})^7\big)$$

and $\lambda_0^-(\varepsilon) = \overline{\lambda_0^+(\varepsilon)}$. These two eigenvalues are perturbations of the double (not semisimple) eigenvalue $\lambda_0(0) = 0$ of $A_\alpha(0)$.

Coming back to the dimensional variables, we obtain from Theorem 3 the following result.

Corollary 4. *The solutions of the partial differential equation problem (2) when m is large can be approximated when $t \to \infty$ by the solutions of the limit ODE*

$$mw''(t) + k_1 w'(t) + k_0 w(t) = 0, \tag{13}$$

where

$$k_1 = \left(\frac{E_1}{L}+q\right) - \frac{1}{3}\left(\frac{E_1}{L}+q\right)\left(\frac{\rho L}{m}\right) + \left(\frac{4E_1}{45L}+\frac{q}{15}\right)\left(\frac{\rho L}{m}\right)^2 + \cdots,$$

$$k_0 = \frac{E}{L}\left[1 - \frac{1}{3}\left(\frac{\rho L}{m}\right) + \frac{4}{45}\left(\frac{\rho L}{m}\right)^2 + \left(\frac{q^2}{45E\rho} - \frac{16}{945}\right)\left(\frac{\rho L}{m}\right)^3 + \cdots\right].$$

The solution $w(t)$ can be interpreted as an approximation of $u(L,t)$.

Proof. Theorem 3 gives an approximation for the dominant eigenvalues $\lambda_0^+(\varepsilon)$ and $\lambda_0^-(\varepsilon)$. Then the corresponding second order ODE can be derived and switching back to the dimensional variables we obtain Eq. (13).

If we want to interpret its solutions, we can denote by $v_r(x)$ and $v_i(x)$ the real and the imaginary parts of the corresponding PDE eigenfunctions, obtaining then the following approximation of the solution for the PDE problem (2):

$$u(x,t) \simeq A(t)v_r(x) + B(t)v_i(x),$$

where $A(t)$ and $B(t)$ turn out to be solutions of the ODE (13). That is why the solution $w(t) \equiv A(t)v_r(L) + B(t)v_i(L)$ has a sense as an approximation of $u(L,t)$. $\square$

Remark 5. We will only prove the existence of two dominant eigenvalues when ε is small enough, but we believe that this will be the most frequent behavior for other values of the parameters. In spite of this, there can be some cases in which the minimum set of dominant eigenvalues is formed by four (or more) eigenvalues and then the system would not be approximable by a second order ODE, as the classical model (1). It could also happen that the dominant part of the spectrum would be the essential spectrum or that in some limit case would not exist a finite dominant part. The analysis of these situations would not be done in the present paper (see [14]).

The proof of Theorem 3 is done at the end of this section, as we need first to prove three lemmas concerning the essential spectra, a uniform bound for the spectra and the convergence in the generalized sense of $A_\alpha(\varepsilon)$ to $A_\alpha(0)$ as $\varepsilon \to 0$ (Lemmas 7–9).

The first thing is to look at the essential spectra of the operators in the sense of the following definition (see [8, §5] and [6]).

Definition 6. Let L be a lineal operator in a Banach space X. We say that λ is a *normal point of* L if λ belongs to the resolvent set of L ($\lambda \in \rho(L)$) or if λ is an isolated eigenvalue of L with finite algebraic multiplicity. Otherwise we say λ belong to the *essential spectrum of* L ($\lambda \in \sigma_{\text{ess}}(L)$). We write $\sigma_p(L) = \sigma(L) \setminus \sigma_{\text{ess}}(L)$.

The essential spectrum as it is defined in Definition 6 turns out to be very stable under relatively compact perturbations. Using this, we can prove the following result.

Lemma 7 (Essential spectra). *The essential spectrum of the operator* $(A_\alpha(\varepsilon), \mathcal{D}(A_\alpha(\varepsilon)))$ *for* $\alpha > 0$ *and* $\varepsilon \geqslant 0$ *is*

$$\sigma_{\text{ess}}\big(A_\alpha(\varepsilon)\big) = \left\{\frac{-1}{\alpha}\right\}.$$

Proof. We first consider the operator for $\varepsilon = 0$, that is $A_\alpha(0)$. And we consider the following relatively compact perturbation of $A_\alpha(0)$:

$$\big(A_\alpha(0) + B\big)\begin{pmatrix}(u, u(1))\\(v, v(1))\end{pmatrix} = \begin{pmatrix}(v, v(1))\\((u+\alpha v)_{xx}, 0)\end{pmatrix} + \begin{pmatrix}(0,0)\\\big(-\frac{1}{\alpha}v, 0\big)\end{pmatrix}$$

whose essential spectrum is

$$\sigma_{\mathrm{ess}}\big(A_\alpha(0)+B\big)=\left\{\frac{-1}{\alpha}\right\}.$$

This can be proved following the same idea as in [3]: under a natural change of variables, the operator can be written in the form

$$\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix},$$

where

$$T_1=\begin{pmatrix} \alpha\partial_x^2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad T_2=\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T_3=\begin{pmatrix} -\frac{1}{\alpha}\mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}$$

whose essential spectrum can be easily calculated and is $\{-1/\alpha\}$. As $A_\alpha(0)$ is a relatively compact perturbation of $A_\alpha(0)+B$, it can be proved that its essential spectrum is the same (see [14]).

So $\sigma_{\mathrm{ess}}(A_\alpha(0))=\{-1/\alpha\}$. For $\varepsilon>0$, the only thing we have to prove is that $A_\alpha(\varepsilon)$ is a relatively compact perturbation of $A_\alpha(0)$. Then one can repeat the arguments above and obtain

$$\sigma_{\mathrm{ess}}\big(A_\alpha(\varepsilon)\big)=\left\{\frac{-1}{\alpha}\right\}. \qquad \square$$

To prove that $\{\lambda_0^+(\varepsilon),\lambda_0^-(\varepsilon)\}$ is a subset of dominant eigenvalues for small positive ε, we also use the following lemma.

Lemma 8 (Spectra uniform bound). *The spectrum of the operators $A_\alpha(\varepsilon)$, $\forall\varepsilon>0$, $\forall\alpha>0$, is contained in the following parabolic sector* (*depending on α but not on ε*):

$$S_\alpha=\left\{x+iy\in\mathbb{C},\ |y|\leqslant 2\sqrt{\frac{-x}{\alpha}},\ x\leqslant 0\right\}.$$

Proof. The numerical range of an operator T in a Hilbert space X is defined as

$$\Theta(T)=\big\{\langle Tu,\ u\rangle\in\mathbb{C},\ u\in\mathcal{D}(T),\ \|u\|_X=1\big\}$$

and under certain hypothesis it can be proved that $\sigma(T)\subset\overline{\Theta(T)}$ (see [9]). Simply using the special inner product of $\mathcal{H}$ defined in Section 3, we can see that

$$\Theta\big(A_\alpha(\varepsilon)\big)\subset S_\alpha,\quad \forall\varepsilon>0,\ \forall\alpha>0.$$

And to prove that $\sigma(A_\alpha(\varepsilon))\subset\overline{\Theta}(A_\alpha(\varepsilon))\subset S_\alpha$ we only need to see that $R(A_\alpha(\varepsilon)-\mathrm{Id})=\mathcal{H}$, that is, the deficiency index of $A_\alpha(\varepsilon)$ is 0 (see [14] for details). $\square$

For proving that perturbed eigenvalues are near the nonperturbed ones, we also need the concept of generalized convergence (again from [9]), which essentially is the convergence between the graphs of the operators. The main idea of these perturbation results is that if the distance (in the generalized sense) between two operators is small enough, compact subsets of the spectra will also be close to each other. That is what is going to happen in our system when $\varepsilon\to 0$.

Lemma 9 (Generalized convergence of operators). *$A_\alpha(\varepsilon)$ converges in the generalized sense to $A_\alpha(0)$ when $\varepsilon \to 0$, for a fixed $\alpha > 0$.*

Proof. We can see that we only have to check that

$$\left\| \left(A_\alpha(\varepsilon) - \mathrm{Id}\right)^{-1} - \left(A_\alpha(0) - \mathrm{Id}\right)^{-1} \right\|_{\mathcal{H}} \underset{\varepsilon \to 0}{\longrightarrow} 0$$

(see [14]). □

Proof of Theorem 3. The eigenvalues for $A_\alpha(0)$ and $\alpha > 0$ can be easily calculated:

$$\lambda_n = \frac{-\alpha\pi^2 n^2 \pm \sqrt{\alpha^2\pi^4 n^4 - 4\pi^2 n^2}}{2}, \quad n = 0, 1, 2, 3, \ldots.$$

We can observe that they are all simple except for $\lambda_0 = 0$ which turns out to be a double and dominant eigenvalue because $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \sigma(A_\alpha(0)) \setminus \{0\}$. We can also observe that the essential spectrum, $-1/\alpha$, is the limit of an eigenvalues subset. For the eigenvalues of $A_\alpha(\varepsilon)$, $\varepsilon > 0$, the following characteristic equation can be derived if we look for solutions of (2) of the form $u(x,t) = e^{\lambda t} u(x)$:

$$\begin{aligned} &[\lambda + \varepsilon\sqrt{1+\lambda\alpha} + \varepsilon r] \exp\left(\frac{\lambda}{\sqrt{1+\lambda\alpha}}\right) \\ &\quad = [\lambda - \varepsilon\sqrt{1+\lambda\alpha} + \varepsilon r] \exp\left(\frac{-\lambda}{\sqrt{1+\lambda\alpha}}\right). \end{aligned} \tag{14}$$

Using the implicit function theorem, we can write λ as a power series of $i\sqrt{\varepsilon}$. Then we can see that for small values of ε the nonperturbed double eigenvalue $\lambda_0(0) = \lambda_0 = 0$ splits into two complex conjugate eigenvalues, denoted as $\lambda_0^+(\varepsilon)$ and $\lambda_0^-(\varepsilon)$. These perturbations for small ε can be analytically approximated, obtaining the formulas given in Theorem 3 (see [14] for details).

What we would expect for $A_\alpha(\varepsilon)$, $\varepsilon > 0$, is that the perturbation of $\lambda_0 = 0$ would keep its dominance, at least if ε is *small enough*. To prove this, we only have to find an appropriate compact set R_α (see Fig. 2) such that the only eigenvalue of $A_\alpha(0)$ that it encloses is $\lambda_0(0) = 0$, with a height greater than the sector S_α's height and such that the essential spectrum $\{-1/\alpha\}$ is outside it.

With such a compact set we have the following. The first thing is that if ε is small enough, $A_\alpha(\varepsilon)$ has the same number of eigenvalues as $A_\alpha(0)$ inside the compact R_α, taking algebraic multiplicities into account (because of Lemma 9). As $\lambda_0(0) = 0$ is a double eigenvalue, $A_\alpha(\varepsilon)$ only has two eigenvalues in R_α if $\varepsilon < \varepsilon_0$, which are $\lambda_0^+(\varepsilon)$ and $\lambda_0^-(\varepsilon)$. This ε_0 depends on the limit operator and on R_α, that is, depends on α and r.

The other thing we have is that the rest of eigenvalues of $A_\alpha(\varepsilon)$ are bounded, in the sense that they are inside the same S_α for all ε which is a sector not depending on ε. And as R_α is taller than this sector, these eigenvalues have a real part minor than $\mathrm{Re}(\lambda_0^+(\varepsilon)) = \mathrm{Re}(\lambda_0^-(\varepsilon))$ (recall they are in $S_\alpha \setminus (R_\alpha \cap S_\alpha)$).

So the only eigenvalues for $A_\alpha(\varepsilon)$ in this region are $\lambda_0^+(\varepsilon)$ and $\lambda_0^-(\varepsilon)$ if ε is small enough. □

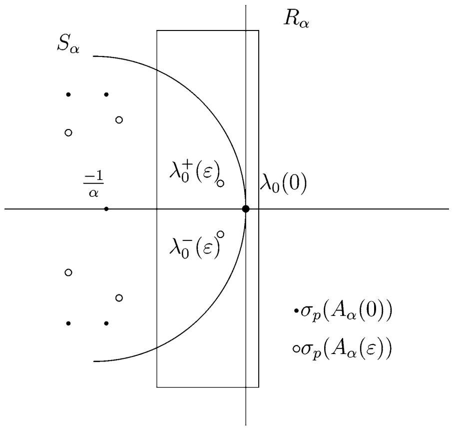

Fig. 2. Scheme for the proof of Theorem 3.

References

[1] C. Castro, E. Zuazua, Boundary controllability of a hybrid system consisting in two flexible beams connected by a point mass, SIAM J. Control Optim. 36 (1998) 1576–1595.
[2] S. Chen, K. Liu, Z. Liu, Spectrum and stability for elastic systems with global or local Kelvin–Voigt damping, SIAM J. Appl. Math. 59 (1999) 651–668.
[3] N. Cónsul, J. Solà-Morales, Stability of local minima and stable nonconstant equilibria, J. Differential Equations 157 (1999) 61–81.
[4] J.L. Davis, Mathematics of Wave Propagation, Princeton Univ. Press, 2000.
[5] K.-J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer, 2000.
[6] I.C. Gohberg, M.G. Kreĭn, Introduction to the Theory of Linear Nonselfadjoint Operators in a Hilbert Space, American Mathematical Society, 1991.
[7] M. Grobbelaar-van Dalsen, On fractional powers of a closed pair of operators and a damped wave equation with dynamic boundary conditions, Appl. Anal. 53 (1994) 41–54.
[8] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Springer, 1989.
[9] T. Kato, Perturbation Theory for Linear Operators, Springer, 1980.
[10] K. Liu, Z. Liu, Exponential decay of energy of Euler–Bernoulli beam with locally distributed Kelvin–Voigt damping, SIAM J. Control Optim. 36 (1998) 1086–1098.
[11] A. Freiria Neves, H. de Souza Ribeiro, O. Lopes, On the spectrum of evolution operators generated by hyperbolic systems, J. Functional Anal. 67 (1986) 320–344.
[12] P. Massat, Limiting behaviour for strongly damped nonlinear wave equations, J. Differential Equations 48 (1983) 334–349.
[13] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, 1983.
[14] M. Pellicer, Ph.D. thesis, Universitat Politècnica de Catalunya, 2004, in preparation.
[15] M. Renardy, On the type of certain C_0-semigroups, Comm. Partial Differential Equations 18 (1993) 1299–1307.