J. Differential Equations 246 (2009) 2813–2828

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Optimal decay rates and the selfadjoint property in overdamped systems ☆

M. Pellicer [a,*], J. Solà-Morales [b]

[a] Dpt. Informàtica i Matemàtica Aplicada, Universitat de Girona, Building P-IV, Campus Montilivi, 17071 Girona, Catalonia, Spain
[b] Dpt. Matemàtica Aplicada 1, Universitat Politècnica de Catalunya, Avda. Diagonal 647, 08028 Barcelona, Catalonia, Spain

ARTICLE INFO

Article history:
Received 2 June 2008
Revised 2 September 2008
Available online 4 February 2009

Keywords:
Optimal decay rate
Selfadjoint operators
Overdamping
Strongly damped wave equations

ABSTRACT

We deal with abstract linear strongly damped wave equations. In the so-called overdamped regime we show the occurrence of two interesting phenomena. The first is the existence of an explicit special inner product which makes the problem selfadjoint. The second is an improvement of the decay rate for more regular solutions that will be of an exponential–polynomial type. Furthermore, we prove the optimality of this decay rate.

1. Introduction

In the recent papers [11,12] and [10] we have studied the large time behaviour of solutions for a mechanical system modelled by a strongly damped wave equation. During this study, we have become acquainted with the interest of the so-called overdamped regime and therefore have started to study the influence of this regime in the decay rate of the solutions. The interest in these decay rates also arises in Control Theory: they provide us with a notion about the velocity at which the system can be controlled. In other words, the decay rate can be viewed as a rate of stabilization of the system.

Let us consider, for example, the strongly damped wave equation in a bounded domain $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary conditions, that is:

$$\begin{cases} u_{tt} - \alpha \Delta u_t - \Delta u = 0 & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases} \tag{1}$$

☆ Partially supported by MEC, Spain (MTM2005-07660-C02-02 and MTM2005-07660-C02-01).
* Corresponding author.
E-mail addresses: martap@ima.udg.edu (M. Pellicer), jc.sola-morales@upc.es (J. Solà-Morales).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2009.01.010

The parameter $\alpha > 0$ measures the strong internal damping of the system. The spectral analysis of (1) shows two interesting phenomena. First, the spectrum has two sequences of eigenvalues, one tending to $-\infty$ and the other accumulating at a finite point $-\omega < 0$, which is the only point in the essential spectrum. Second, when α is large enough the whole spectrum becomes real.

This second phenomenon suggests the question of the existence or not of an inner product in which the corresponding operator is selfadjoint. On the other hand, although it is known that all the solutions decay as $e^{-\omega t}$, the fact that $\lambda = -\omega$ is an accumulation point of the eigenvalues but it is not an eigenvalue by itself may cause a better decay rate, like $e^{-\omega t}/t^{\gamma}$, for certain dense classes of initial conditions. The goal of the present paper is to give an answer to these two questions. This will be done not only for the strongly damped wave equation but for an abstract equation including this case, namely:

$$u_{tt} + \alpha B u_t + Au = 0. \tag{2}$$

Here A, B represent operators in a Hilbert space $\mathcal{H}$ that should commute with each other. The answer to these questions is affirmative essentially when the damping is large enough, as we will see below. This is known as the overdamped regime. Roughly speaking, it occurs when the damping in the equation is so high that it makes all the oscillations to disappear and also, contradicting intuition, it even causes a slower rate of decay on the solutions.

This is a very elementary and well-known fact for the simple ordinary differential equation:

$$x'' + \alpha x' + kx = 0$$

($k > 0$). It occurs when $\alpha > 2\sqrt{|k|}$. One can write this equation as a first order system of the form

$$\frac{d}{dt}\begin{pmatrix} x \\ x' \end{pmatrix} = L \begin{pmatrix} x \\ x' \end{pmatrix} \tag{3}$$

and then construct a new scalar product in which L becomes a symmetric matrix. This construction in dimension two has been our source of inspiration for the definition of the abstract scalar product that is presented below. Writing Eq. (2) as a first order system of the same type as (3), we obtain the following result.

Theorem 1 *(L selfadjoint). Let $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{D}(B) \subset \mathcal{H} \to \mathcal{H}$ be selfadjoint operators with compact resolvent in a Hilbert space $\mathcal{H}$, with $\mathcal{D}(A) \subset \mathcal{D}(B) \subset \mathcal{D}(A^{1/2})$ continuously and satisfying the following hypotheses*:

(H1) $$\begin{cases} (Au, u)_{\mathcal{H}} \geqslant \nu (u, u)_{\mathcal{H}} & \text{for all } u \in \mathcal{D}(A), \\ (Bu, u)_{\mathcal{H}} \geqslant \nu' (u, u)_{\mathcal{H}} & \text{for all } u \in \mathcal{D}(B), \end{cases}$$

for some $\nu, \nu' > 0$ (strictly positive *operators*).

(H2) $(Au_1, Bu_2)_{\mathcal{H}} = (Bu_1, Au_2)_{\mathcal{H}}$ *for all $u_1, u_2 \in \mathcal{D}(A)$* (commutativity *condition*).

(H3) $\alpha > 2M$ (*the* overdamping *condition*), *where M is the relative bound $\|A^{1/2}u\|_{\mathcal{H}} \leqslant M \|Bu\|_{\mathcal{H}}$ corresponding to the continuous embedding $\mathcal{D}(B) \subset \mathcal{D}(A^{1/2})$.*

Then, the operator $L : \mathcal{D}(L) = \mathcal{D}(A) \times \mathcal{D}(B) \subset X \to X = \mathcal{D}(B) \times \mathcal{H}$ defined as

$$\begin{pmatrix} 0 & I \\ -A & -\alpha B \end{pmatrix}$$

is selfadjoint with the following inner product $(\,,\,)_E$, equivalent to the natural inner product of X:

$$\left(\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}\right)_E := \frac{\alpha^2}{2}(Bu_1, Bu_2)_{\mathcal{H}} - \left(A^{\frac{1}{2}}u_1, A^{\frac{1}{2}}u_2\right)_{\mathcal{H}} + \frac{\alpha}{2}(Bu_1, v_2)_{\mathcal{H}} + \frac{\alpha}{2}(v_1, Bu_2)_{\mathcal{H}} + (v_1, v_2)_{\mathcal{H}}. \tag{4}$$

Moreover, the operator L is dissipative with this scalar product $(\,,\,)_E$, that is:

$$\left(L\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right)_E \leqslant 0 \quad \textit{for all } \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(L). \tag{5}$$

Let us comment on some aspects of this special inner product, which can be formally written as

$$\left(\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}\right)_E = (u_1, v_1)\begin{pmatrix} \frac{\alpha^2}{2}B^2 - A & \frac{\alpha}{2}B \\ \frac{\alpha}{2}B & \mathrm{Id} \end{pmatrix}\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}.$$

Observe that when $B = A$ hypothesis (H2) automatically holds and that for the overdamping condition (H3) to be fulfilled $\alpha > 2/\sqrt{\nu}$ is required. Also, if we consider Eq. (1) with Ω a bounded domain in $\mathbb{R}^n$ with a smooth boundary and $\mathcal{H} = L^2(\Omega)$, the quadratic form associated with this new scalar product $(\,,\,)_E$ can be written more explicitly:

$$\left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\|_E^2 = \frac{\alpha^2}{2}\int_\Omega |\Delta u|^2 - \int_\Omega |\nabla u|^2 - \frac{\alpha}{2}\int_\Omega (\Delta u)\bar{v} - \frac{\alpha}{2}\int_\Omega v(\Delta\bar{u}) + \int_\Omega |v|^2.$$

To our knowledge, the operator corresponding to an equation of type (2) in the overdamped regime has never before been shown to be selfadjoint in a suitable scalar product. The fact that in the overdamped case the spectrum of an equation of this type is real was already observed by [18] and [3].

The central point of the present paper is precisely the existence of this scalar product, in which the operator L is selfadjoint. With this we will be able to work in the corresponding orthonormal basis in order to obtain sharp decay inequalities. This situation recalls that of the work in [21], where this property was also used to obtain a result of unique continuation for a Benjamin–Bona–Mahony equation.

Remark 2. Observe that under the hypotheses of Theorem 1 the operators A and B have a common countable basis $\{\varphi_n\}$ of orthonormal eigenfunctions. To see this, we first note that A and B are invertible operators and that A^{-1} and B^{-1} are bounded, selfadjoint and compact operators in $\mathcal{H}$. Also, by writing (H2) for $v_1 = Au_1$ and $v_2 = Au_2$ one obtains

$$\left(v_1, BA^{-1}v_2\right)_{\mathcal{H}} = \left(BA^{-1}v_1, BA^{-1}v_2\right)_{\mathcal{H}}$$

for all $v_1, v_2 \in \mathcal{H}$. In particular, for $v_1, v_2 \in \mathcal{D}(B)$ we can change variables again to $w_1 = Bv_1$ and $w_2 = Bv_2$ and get

$$\left(B^{-1}w_1, BA^{-1}B^{-1}w_2\right)_{\mathcal{H}} = \left(BA^{-1}B^{-1}w_1, B^{-1}w_2\right)_{\mathcal{H}}$$

for all $w_1, w_2 \in \mathcal{H}$. Now that B^{-1} is selfadjoint, one can use it to obtain

$$\left(w_1, A^{-1}B^{-1}w_2\right)_{\mathcal{H}} = \left(A^{-1}B^{-1}w_1, w_2\right)_{\mathcal{H}} = \left(w_1, B^{-1}A^{-1}w_2\right)_{\mathcal{H}}$$

for all $w_1, w_2 \in \mathcal{H}$. Therefore, $A^{-1}B^{-1} = B^{-1}A^{-1}$, that is, A^{-1} and B^{-1} commute.

Applying Theorem 9.7 on Chapter 9.2 of [4], we can conclude that A^{-1} and B^{-1} have a common basis $\{\varphi_n\}$ of orthonormal eigenfunctions. This is the common basis for A and B we were looking for.

Remark 3. The property of L being selfadjoint together with the inequality (5) immediately implies the analyticity of the semigroup e^{Lt}. Outside the overdamped regime, the analyticity of this semigroup can also be proved by observing that L is a relatively bounded perturbation of another operator L' that is in the overdamped regime. This is not an especially new result (see [2,9], for example) but it is worth mentioning.

The fact of L being a selfadjoint operator and inequality (5) give the existence of an optimal constant $\omega \geqslant 0$ such that

$$\left\| e^{Lt} \right\|_E \leqslant e^{-\omega t}.$$

On the other hand, we are concerned with the question of whether a better decay rate is possible for the solutions of (2). We will show that it is true when $B = A$ and in the overdamped regime for certain classes of smoother initial conditions (that will be dense in the whole space).

Definition 4. (See [8].) Let us consider a positive function $f(t)$ such that $\lim_{t\to\infty} f(t) = 0$. If $\vec{U}(t) = e^{Lt}\vec{U}(0)$ and

$$\left\| \vec{U}(t) \right\|_X \leqslant f(t) \cdot \left\| \vec{U}(0) \right\|_D, \quad t > 0,$$

for a certain subspace $D \subset X$, we say that $\vec{U}(t)$ *decays at the rate of* $f(t)$.

Definition 5. We say that the decay rate of $\vec{U}(t)$ is *exponential–polynomial* if

$$f(t) = \frac{e^{-\omega t}}{t^\gamma}$$

for some ω, $\gamma \geqslant 0$.

The result in our case is the following.

Theorem 6 *(Decay rate of solutions). Let us consider the solutions of* (2) *with $B = A$ such that the hypotheses of Theorem* 1 *are satisfied. In this case, one has*

$$\left\| e^{Lt} \right\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \leqslant e^{-\omega t} \tag{6}$$

where $\omega = 1/\alpha$ is the optimal possible value for ω. Then, for each $\gamma \geqslant 0$, there exists a constant $K > 0$ such that

$$\left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \right\|_E \leqslant K \cdot \left\| \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \right\|_{R_\gamma} \cdot \frac{e^{-\frac{1}{\alpha}t}}{t^{\frac{\gamma}{2}}} \tag{7}$$

for all $\binom{u(0)}{v(0)} \in R_\gamma$, $\gamma \geqslant 0$, defined as

$$R_\gamma = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(A) \times \mathcal{H},\ u \in \mathcal{D}\big(A^{\gamma/2}\big),\ \alpha A u + v \in \mathcal{D}\big(A^{\gamma/2}\big) \right\} \tag{8}$$

and

$$\left\| \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \right\|_{R_\gamma} = \left\| u(0) \right\|_{\mathcal{D}(A)} + \left\| v(0) \right\|_{\mathcal{H}} + \left\| u(0) \right\|_{\mathcal{D}(A^{\gamma/2})} + \left\| \alpha A u(0) + v(0) \right\|_{\mathcal{D}(A^{\gamma/2})}.$$

This is our result on exponential–polynomial decay. The spectrum of L consists of a sequence of eigenvalues $\mu_n^- \to -\infty$, another sequence $\mu_n^+ \to -1/\alpha$ (with $\mu_n^+ < -1/\alpha$ for all $n \in \mathbb{N}$) and a single point $-1/\alpha$, which is not an eigenvalue but the only point in the essential spectrum of L. Roughly speaking, the fact that $-1/\alpha$ is not an eigenvalue gives us the chance to improve (6) for smoother solutions.

Although (6) gives an exponential decay, this decay is not very strong when α is large. Therefore, even a slight improvement in this rate, as the one given in (7) for initial conditions in the dense subset R_γ, is worth showing. Actually, we are able to prove that, for those sets, the decay is optimal in the following sense.

Theorem 7 *(Optimality of the decay rate). Under the same hypotheses as Theorem* 6*, let us consider the spaces R_γ, $\gamma \geqslant 0$, defined in* (8)*. Then, the decay rate given in Theorem* 6 *is optimal in the following sense*: *it is not possible to find $G : R_\gamma \to [0,\infty)$ (even not continuous) and a function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(t) \to 0$ as $t \to \infty$ such that*

$$\left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \right\|_E \leqslant G\left(\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \right) \frac{e^{-\frac{1}{\alpha}t}}{t^{\frac{\gamma}{2}}} \phi(t)$$

for all $t \geqslant 0$ and all $\binom{u(0)}{v(0)} \in R_\gamma$.

To our knowledge, decay rates of the exponential–polynomial type have appeared in the literature only when $\omega = 0$. There are several methods to obtain this type of decay, and an extensive literature about them, starting with [18]. Our method for obtaining the decay rate is somehow near the methods in the literature that use appropriate Riesz bases, like [7,22,23]. It must also be said that the optimality of the polynomial decay rate has been stated in the last two papers. The *Riesz basis* property in our case comes from the selfadjoint property for the linear operator. Riesz bases adapted to other similar strongly damped wave equations are obtained in [19] or [20].

Other methods to obtain this type of decay are based on observability inequalities, with origins in control theory or in energy estimates. Some examples can be seen in [6,15,16] or [17]. These methods have also been applied for a wave equation but only in the case of weak damping (see [13] as an example). Estimates for the resolvent operator and spectral analysis are the techniques used in [8] and [1]. The recent work [14] also uses this approach.

The results we have given in this section are proved in the following two sections. Section 2 studies the new scalar product $(\,,\,)_E$ and the selfadjointness of the operator L respect to it, as it has been stated in Theorem 1. In Section 3, the case $A = B$ is treated and we show the exponential–polynomial decay rate for solutions with sufficiently smooth initial conditions (Theorem 6). We also prove in this section the optimality of this decay rate, as Theorem 7 claims.

2. The selfadjoint property

The aim of this section is to prove Theorem 1. The first thing is to guarantee that the expression given in (4) corresponds to a well-defined inner product.

Lemma 8. *If the overdamping condition* (H3) *holds, then the product defined in* (4) *is a well-defined inner product in $X = \mathcal{D}(B) \times \mathcal{H}$ and it is equivalent to its natural inner product.*

Proof. As $(\,,\,)_{\mathcal{H}}$ is an inner product in $\mathcal{H}$, the only thing that needs to be checked is the fact of $(\,,\,)_E$ being positively defined. That is, we have to check if:

$$\left(\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right)_E > 0 \quad \text{for all } \begin{pmatrix} u \\ v \end{pmatrix} \in X = \mathcal{D}(B) \times \mathcal{H}, \quad \begin{pmatrix} u \\ v \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By writing the previous expression, we have

$$\left(\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right)_E = \frac{\alpha^2}{2}\|Bu\|_{\mathcal{H}}^2 - \left\|A^{\frac{1}{2}}u\right\|_{\mathcal{H}}^2 + \alpha \operatorname{Re}(Bu, v)_{\mathcal{H}} + \|v\|_{\mathcal{H}}^2. \tag{9}$$

By Schwarz inequality and adding and subtracting $\varepsilon\|v\|_{\mathcal{H}}^2$, $0 < \varepsilon < 1$, we have

$$(9) \geqslant \frac{\alpha^2}{2}\|Bu\|_{\mathcal{H}}^2 - \left\|A^{\frac{1}{2}}u\right\|_{\mathcal{H}}^2 - \alpha\|Bu\|_{\mathcal{H}}\|v\|_{\mathcal{H}} + (1-\varepsilon)\|v\|_{\mathcal{H}}^2 + \varepsilon\|v\|_{\mathcal{H}}^2.$$

The previous expression can be written in the following way:

$$(9) \geqslant \left(\frac{\alpha^2}{2} - \frac{\alpha^2}{4(1-\varepsilon)}\right)\|Bu\|_{\mathcal{H}}^2 - \left\|A^{\frac{1}{2}}u\right\|_{\mathcal{H}}^2 + \left(\frac{\alpha}{2\sqrt{1-\varepsilon}}\|Bu\|_{\mathcal{H}} - \sqrt{1-\varepsilon}\|v\|_{\mathcal{H}}\right)^2 + \varepsilon\|v\|_{\mathcal{H}}^2.$$

Adding and subtracting $\varepsilon\|Bu\|_{\mathcal{H}}^2$ to the right-hand side of the previous inequality, we obtain

$$\left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\|_E^2 \geqslant \left(\frac{\alpha^2}{2} - \frac{\alpha^2}{4(1-\varepsilon)} - \varepsilon\right)\|Bu\|_{\mathcal{H}}^2 - \left\|A^{\frac{1}{2}}u\right\|_{\mathcal{H}}^2 + \left(\frac{\alpha}{2\sqrt{1-\varepsilon}}\|Bu\|_{\mathcal{H}} - \sqrt{1-\varepsilon}\|v\|_{\mathcal{H}}\right)^2 + \varepsilon\|Bu\|_{\mathcal{H}}^2 + \varepsilon\|v\|_{\mathcal{H}}^2.$$

Therefore, the lemma holds if

$$\left(\frac{\alpha^2}{2} - \frac{\alpha^2}{4(1-\varepsilon)} - \varepsilon\right)\|Bu\|_{\mathcal{H}}^2 - \left\|A^{\frac{1}{2}}u\right\|_{\mathcal{H}}^2 \geqslant 0$$

for some $\varepsilon \in (0,1)$. As $\mathcal{D}(B)$ is continuously embedded in $\mathcal{D}(A^{1/2})$ we know that

$$M^2\|Bu\|_{\mathcal{H}}^2 \geqslant \left\|A^{\frac{1}{2}}u\right\|_{\mathcal{H}}^2$$

for a certain $M > 0$. Then, we have to find $\varepsilon \in (0,1)$ such that

$$r(\varepsilon) := \left(\frac{\alpha^2}{2} - \frac{\alpha^2}{4(1-\varepsilon)} - \varepsilon\right) > M^2. \tag{10}$$

Observe that by hypothesis (H3) we know that

$$r(0) = \frac{\alpha^2}{4} > \frac{4M^2}{4} = M^2.$$

Hence, we can assure by continuity arguments that (10) holds for ε small enough. Therefore, for such an ε it is satisfied that

$$\left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\|_E^2 \geqslant \varepsilon\left(\|Bu\|_{\mathcal{H}}^2 + \|v\|_{\mathcal{H}}^2\right) = \varepsilon\left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\|_X^2. \tag{11}$$

The opposite inequality can easily be seen:

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_E^2 \leqslant \frac{\alpha^2}{2} \|Bu\|_{\mathcal{H}}^2 + \|Bu\|_{\mathcal{H}} \|v\|_{\mathcal{H}} + \|v\|_{\mathcal{H}}^2$$
$$\leqslant C\big(\|Bu\|_{\mathcal{H}}^2 + \|v\|_{\mathcal{H}}^2\big) = C \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_X^2 \tag{12}$$

for a certain $C > 0$. Inequalities (11) and (12) conclude that $(\,,)_E$ is positively defined, defines an inner product in $X = \mathcal{D}(B) \times \mathcal{H}$ and it is equivalent to the natural one in X. □

We can now prove Theorem 1.

Proof of Theorem 1. The first part of the theorem states the selfadjointness of the operator L with the inner product $(\,,)_E$ defined in (4). To prove this selfadjointness we will see that L is symmetric with $(\,,)_E$ and that L is an invertible operator in X.

(a) *L is symmetric with $(\,,)_E$.* For $\binom{u_1}{v_1}, \binom{u_2}{v_2} \in \mathcal{D}(L)$ and using the definition of $(\,,)_E$ given in (4) it can be seen that

$$\left(L \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right)_E = -\big(A^{\frac{1}{2}} v_1, A^{\frac{1}{2}} u_2\big)_{\mathcal{H}} - \frac{\alpha}{2}(Bv_1, v_2)_{\mathcal{H}} - (Au_1, v_2)_{\mathcal{H}} - \frac{\alpha}{2}(Au_1, Bu_2)_{\mathcal{H}}.$$

As A and B are selfadjoint operators, the previous expression is equal to the following:

$$= -(v_1, Au_2)_{\mathcal{H}} - \frac{\alpha}{2}(v_1, Bv_2)_{\mathcal{H}} - \big(A^{\frac{1}{2}} u_1, A^{\frac{1}{2}} v_2\big)_{\mathcal{H}} - \frac{\alpha}{2}(Au_1, Bu_2)_{\mathcal{H}}$$

where the only thing that remains to be checked for symmetry is the last term. But, as A and B are commuting operators (in the sense of hypothesis (H2) of Theorem 1) we conclude that

$$= -(v_1, Au_2)_{\mathcal{H}} - \frac{\alpha}{2}(v_1, Bv_2)_{\mathcal{H}} - \big(A^{\frac{1}{2}} u_1, A^{\frac{1}{2}} v_2\big)_{\mathcal{H}} - \frac{\alpha}{2}(Bu_1, Au_2)_{\mathcal{H}}$$
$$= \left(\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, L \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right)_E.$$

Therefore, L is symmetric with the inner product $(\,,)_E$.

(b) *L is invertible.* We have to check whether for a given $\binom{f}{g} \in X$ there exists $\binom{u}{v} \in \mathcal{D}(A) \times \mathcal{D}(B)$ such that

$$L \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

that is, such that

$$\begin{pmatrix} v \\ -Au - \alpha Bv \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

As A is strictly positive it is invertible, and the previous equation can be solved in $\mathcal{D}(A) \times \mathcal{D}(B)$.

The second part of the theorem states the dissipativeness of L. To see this, let us write the first part of inequality (5) using the fact that A is a selfadjoint operator:

$$\left(L\begin{pmatrix}u\\v\end{pmatrix},\begin{pmatrix}u\\v\end{pmatrix}\right)_E=\frac{\alpha^2}{2}(Bv,Bu)_{\mathcal{H}}-\left(A^{\frac12}v,A^{\frac12}u\right)_{\mathcal{H}}+\frac{\alpha}{2}(Bv,v)_{\mathcal{H}}$$
$$+\frac{\alpha}{2}(-Au-\alpha Bv,Bu)_{\mathcal{H}}+(-Au-\alpha Bv,v)_{\mathcal{H}}$$
$$=-2\operatorname{Re}(Au,v)_{\mathcal{H}}-\frac{\alpha}{2}(Bv,v)_{\mathcal{H}}-\frac{\alpha}{2}(Au,Bu)_{\mathcal{H}}.$$

To see that the previous expression is non-positive, we are going to prove that

$$\left(\operatorname{Re}(Au,v)_{\mathcal{H}}\right)^2-\frac{\alpha^2}{4}(Bv,v)_{\mathcal{H}}\cdot(Au,Bu)_{\mathcal{H}}\leqslant 0. \tag{13}$$

This suffices as (13) is the discriminant of the following equation in x:

$$\frac{\alpha}{2}(Au,Bu)_{\mathcal{H}}x^2+2\operatorname{Re}(Au,v)_{\mathcal{H}}x+\frac{\alpha}{2}(Bv,v)_{\mathcal{H}}=0$$

(that is a positive expression when $x=0$ due to the positiveness of B). Condition (13) can be checked using the following lemma.

Lemma 9. *Under the hypotheses of Theorem* 1, *the following assertions are true*:

(1) *If* $\|A^{\frac12}u\|_{\mathcal{H}}\leqslant M\|Bu\|_{\mathcal{H}}$, *then* $(A^{\frac12}u,u)_{\mathcal{H}}\leqslant M(Bu,u)_{\mathcal{H}}$.
(2) $(A^{\frac12}u_1,Bu_2)_{\mathcal{H}}=(Bu_1,A^{\frac12}u_2)_{\mathcal{H}}$ *for all* $u_1,u_2\in\mathcal{D}(B)$.

Proof. From Remark 2 we know the existence of a common orthonormal basis of eigenfunctions $\{\varphi_n\}$ for A and B: $A\varphi_n=\lambda_n^A\varphi_n$ and $B\varphi_n=\lambda_n^B\varphi_n$. If the inequality $\|A^{\frac12}u\|_{\mathcal{H}}\leqslant M\|Bu\|_{\mathcal{H}}$ is written in this basis, we will see that $(\lambda_n^A)^{1/2}\leqslant\lambda_n^B$ for each $n\in\mathbb{N}$. This automatically implies part (1) of the lemma.

The commutativity condition (H2) for A and B written in this common basis clearly implies the commutativity condition for $A^{\frac12}$ and B, as part (2) of the lemma claims. This concludes the proof of the lemma. □

Let us now come back to the proof of condition (13). It is clear that

$$\left(\operatorname{Re}(Au,v)_{\mathcal{H}}\right)^2\leqslant\left|(Au,v)_{\mathcal{H}}\right|^2. \tag{14}$$

Using the selfadjointness of A and the Schwarz inequality, we obtain that

$$(14)\leqslant\left|\left(A^{3/4}u,A^{1/4}v\right)_{\mathcal{H}}\right|^2$$
$$\leqslant\left\|A^{3/4}u\right\|_{\mathcal{H}}^2\cdot\left\|A^{1/4}v\right\|_{\mathcal{H}}^2=\left(Au,A^{1/2}u\right)_{\mathcal{H}}\cdot\left(A^{1/2}v,v\right)_{\mathcal{H}}.$$

Applying Lemma 9 to both factors of the right-hand side of the previous inequality, we have that

$$\leqslant M\left(B\left(A^{1/2}u\right),A^{1/2}u\right)_{\mathcal{H}}\cdot M(Bv,v)_{\mathcal{H}}.$$

And, by the selfadjointness of B and using Lemma 9, the first factor of the previous expression can be rewritten as

$$\leqslant M^2\left(A^{1/2}u,B\left(A^{1/2}u\right)\right)_{\mathcal{H}}\cdot(Bv,v)_{\mathcal{H}}=M^2(Bu,Au)_{\mathcal{H}}\cdot(Bv,v)_{\mathcal{H}}.$$

As the overdamping condition (H3) is satisfied, we can conclude that

$$\leqslant \frac{\alpha^2}{4}(Bu, Au)_{\mathcal{H}} \cdot M(Bv, v)_{\mathcal{H}}.$$

This completes the proof of condition (13) and, hence, we conclude the dissipativeness of L with the inner product $(\,,\,)_E$. □

3. The decay rate

In this section we study Eq. (2) when $A = B$. This equation can also be written as the first order system

$$\frac{d}{dt}\vec{U}(t) = L\vec{U}(t) \tag{15}$$

where

$$L = \begin{pmatrix} 0 & I \\ -A & -\alpha A \end{pmatrix} : \mathcal{D}(L) = \mathcal{D}(A) \times \mathcal{D}(A) \subset X \to X = \mathcal{D}(A) \times \mathcal{H}.$$

We have seen in Theorem 1 that L is a selfadjoint operator with the inner product $(\,,\,)_E$. As a consequence, the existence of an orthogonal basis in X of eigenfunctions of L is proved. So, the solutions of problem (15) can be expressed in this basis and this fact allows us to analyze the decay rate of these solutions more explicitly.

Let us see some results concerning the spectrum and eigenfunctions of the operator L that will be used later.

Proposition 10. *Suppose that we are under the hypotheses of Theorem* 1 *for the case $A = B$. Then, the following conclusions hold*:

(1) *The eigenvalues of L consist of two sequences $\{\mu_n^-\}$ and $\{\mu_n^+\}$ such that $\mu_n^- \to -\infty$ and $\mu_n^+ \to -\frac{1}{\alpha}$ when $n \to +\infty$. The point $-\frac{1}{\alpha}$ belongs to the spectrum of L, $\sigma(L)$, but it is not an eigenvalue.*
(2) *The eigenfunctions of L are also complete in X.*
(3) *$\sigma(L) \subset (-\infty, -1/\alpha]$.*

Proof. First of all, observe that the eigenfunctions of A are complete in $\mathcal{H}$ and that the spectrum of A, $\sigma(A)$, consists of a sequence $\{\lambda_n\}$ of eigenvalues, with $\lambda_n \to +\infty$.

The eigenvalues and eigenfunctions of L can be described in terms of the eigenvalues and eigenfunctions of A. Indeed, the following equalities are equivalent:

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \mu \begin{pmatrix} u \\ v \end{pmatrix} \quad \Leftrightarrow \quad \begin{cases} v = \mu u, \\ -Au - \alpha A v = \mu v. \end{cases}$$

From here we obtain that the set of eigenvalues of L, $\sigma_p(L)$, consists of

$$\sigma_p(L) = \{\mu^+(\lambda), \mu^-(\lambda) \mid \lambda \in \sigma_p(A)\}$$

where $\mu^{\pm}(\lambda) = (-\lambda\alpha \pm \sqrt{\lambda^2\alpha^2 - 4\lambda})/2$. Observe that for each $\lambda \in \sigma_p(A)$ we have $\mu^+(\lambda) \neq \mu^-(\lambda)$, since hypotheses (H1) and (H3) of Theorem 1 imply that $\lambda^2\alpha^2 - 4\lambda > 0$. Also, if φ is an eigenfunction of A and λ is its associated eigenvalue, then $\binom{\varphi}{\mu^+(\lambda)\varphi}$ and $\binom{\varphi}{\mu^-(\lambda)\varphi}$ are the eigenfunctions of L with associated eigenvalues $\mu^+(\lambda)$ and $\mu^-(\lambda)$, respectively.

With the same calculations one can see that all the eigenfunctions of L have this form and also that $-1/\alpha$ is not an eigenvalue of L.

Consider now the sequence $\{\lambda_n\}$, $n \in \mathbb{N}$, of eigenvalues of A, which tends to $+\infty$ when $n \to \infty$. Defining $\mu_n^+ = \mu^+(\lambda_n)$ and $\mu_n^- = \mu^-(\lambda_n)$ one obtains that $\{\mu_n^\pm\}$ are the eigenvalues of L and that $\mu_n^+ \to -1/\alpha$, with $\mu_n^+ < -1/\alpha$, and $\mu_n^- \to -\infty$. This finishes the proof of part (1) of this proposition.

In order to prove part (2), we first see that the two-dimensional space spanned by the eigenfunctions

$$\begin{pmatrix} \varphi \\ \mu^+(\lambda)\,\varphi \end{pmatrix}, \begin{pmatrix} \varphi \\ \mu^-(\lambda)\,\varphi \end{pmatrix}$$

is the same as the one spanned by

$$\begin{pmatrix} \varphi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi \end{pmatrix}$$

since $\mu^+(\lambda) \neq \mu^-(\lambda)$. Then, to show that the eigenfunctions of L are complete in X we have merely to see that given a vector $\binom{u}{v} \in X$ and $\varepsilon > 0$ there exist $N \in \mathbb{N}$, eigenfunctions $\varphi_1, \varphi_2, \ldots, \varphi_N$ of A and numbers $a_1, a_2, \ldots, a_N$, $b_1, b_2, \ldots, b_N$ such that

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} - \sum_{i=1}^N a_i \begin{pmatrix} \varphi_i \\ 0 \end{pmatrix} - \sum_{j=1}^N b_j \begin{pmatrix} 0 \\ \varphi_j \end{pmatrix} \right\|_X \leqslant \varepsilon.$$

Since $X = \mathcal{D}(A) \times \mathcal{H}$ we know that

$$\begin{aligned} \left\| \begin{pmatrix} u \\ v \end{pmatrix} - \sum_{i=1}^N a_i \begin{pmatrix} \varphi_i \\ 0 \end{pmatrix} - \sum_{j=1}^N b_j \begin{pmatrix} 0 \\ \varphi_j \end{pmatrix} \right\|_X^2 &= \left\| u - \sum_{i=1}^N a_i \varphi_i \right\|_{\mathcal{D}(A)}^2 + \left\| v - \sum_{j=1}^N b_j \varphi_j \right\|_{\mathcal{H}}^2 \\ &= \left\| u - \sum_{i=1}^N a_i A \varphi_i \right\|_{\mathcal{H}}^2 + \left\| v - \sum_{j=1}^N b_j \varphi_j \right\|_{\mathcal{H}}^2 \\ &= \left\| u - \sum_{i=1}^N a_i \lambda_i \varphi_i \right\|_{\mathcal{H}}^2 + \left\| v - \sum_{j=1}^N b_j \varphi_j \right\|_{\mathcal{H}}^2. \end{aligned} \tag{16}$$

Here we have used that the norm $\|Au\|_{\mathcal{H}}$ is equivalent in $\mathcal{D}(A)$ to the usual norm $\|Au\|_{\mathcal{H}} + \|u\|_{\mathcal{H}}$. This is true since A is invertible.

It is clear that the last two terms in (16) can be made as small as desired because the eigenfunctions of A are complete in $\mathcal{H}$. This completes the proof of part (2).

Finally, to see part (3) we use the numerical range of the operator L (see [5]), defined as

$$\Theta(L) = \left\{ (L\vec{U}, \vec{U})_E \in \mathbb{C},\ \text{with } \vec{U} \in \mathcal{D}(L),\ \|\vec{U}\|_E = 1 \right\}.$$

By expanding $\vec{U}$ in terms of the orthonormal basis of X, and using the fact that all the eigenvalues of L are strictly lower than $-1/\alpha$, we can say that

$$\Theta(L) \subset (-\infty, -1/\alpha].$$

Using Theorem 3.2 of Chapter V of [5], the previous inclusion implies that $(-1/\alpha, \infty)$ is a subset of the resolvent set. Therefore, $\sigma(L) \subset (-\infty, -1/\alpha]$.

This completes the proof of the present proposition. □

We can now prove the result on the exponential–polynomial decay for sufficiently regular solutions, given in Theorem 6.

Proof of Theorem 6. In Proposition 10 we have just seen that all the eigenvalues of L are real and strictly lower than $-1/\alpha$. And from Theorem 1 we know that L is selfadjoint with the inner product $(\,,\,)_E$. Then, we can automatically say that any $\vec{U}(t)$ solution of (15) decays exponentially as

$$\|\vec{U}(t)\|_E \leqslant e^{-\frac{1}{\alpha}}\|\vec{U}(0)\|_E.$$

Let us call $\{\vec{e}_n^{\pm}\}$ the basis of orthonormalized eigenfunctions of L with corresponding eigenvalue $\mu_n^{\pm}$, respectively. We can write the initial condition $\vec{U}(0)$ and its corresponding solution of (15), $\vec{U}(t)$, as

$$\vec{U}(0)=\sum_{n=1}^{\infty}a_n\vec{e}_n^{+}+\sum_{n=1}^{\infty}b_n\vec{e}_n^{-},\qquad \vec{U}(t)=\sum_{n=1}^{\infty}a_n e^{\mu_n^{+}t}\vec{e}_n^{+}+\sum_{n=1}^{\infty}b_n e^{\mu_n^{-}t}\vec{e}_n^{-}.$$

We can now compute the $\|\ \|_E$-norm of the solution:

$$\begin{aligned}\|\vec{U}(t)\|_E^2&=\sum_{n=1}^{\infty}|a_n|^2e^{2\mu_n^{+}t}+\sum_{n=1}^{\infty}|b_n|^2e^{2\mu_n^{-}t}\\&=\left(\sum_{n=1}^{\infty}\frac{|a_n|^2}{[-2(\mu_n^{+}+\frac{1}{\alpha})t]^{\gamma}}\left[-2\left(\mu_n^{+}+\frac{1}{\alpha}\right)t\right]^{\gamma}e^{2(\mu_n^{+}+\frac{1}{\alpha})t}\right.\\&\quad\left.+\sum_{n=1}^{\infty}\frac{|b_n|^2}{[-2(\mu_n^{-}+\frac{1}{\alpha})t]^{\gamma}}\left[-2\left(\mu_n^{-}+\frac{1}{\alpha}\right)t\right]^{\gamma}e^{2(\mu_n^{-}+\frac{1}{\alpha})t}\right)e^{-\frac{2}{\alpha}t}.\end{aligned}$$

As $\mu_n^{\pm}<-\dfrac{1}{\alpha}$, observe that $[-2(\mu_n^{\pm}+\frac{1}{\alpha})t]>0$. We now make use of the inequality

$$x^{\gamma}e^{-x}\leqslant\gamma^{\gamma}e^{-\gamma}\quad\text{for any } x>0.$$

Then we obtain

$$\|\vec{U}(t)\|_E^2\leqslant\frac{\gamma^{\gamma}}{e^{\gamma}}\left(\sum_{n=1}^{\infty}\frac{|a_n|^2}{[-2(\mu_n^{+}+\frac{1}{\alpha})]^{\gamma}}+\sum_{n=1}^{\infty}\frac{|b_n|^2}{[-2(\mu_n^{-}+\frac{1}{\alpha})]^{\gamma}}\right)\frac{e^{-\frac{2}{\alpha}t}}{t^{\gamma}}.\tag{17}$$

As $\sum_{n=1}^{\infty}|b_n|^2<\infty$ and $\mu_n^{-}\to-\infty$, the second term of (17) converges:

$$\left(\sum_{n=1}^{\infty}\frac{|b_n|^2}{[-2(\mu_n^{-}+\frac{1}{\alpha})]^{\gamma}}\right)^{\frac{1}{2}}\leqslant C_1\left(\sum_{n=1}^{\infty}|b_n|^2\right)^{\frac{1}{2}}\leqslant C_2\|\vec{U}(0)\|_X<\infty\tag{18}$$

where $C_1, C_2>0$ do not depend on the initial conditions. Remember now that by Proposition 10 we know that $\mu_n^{+}\to-\frac{1}{\alpha}$. Therefore, the first term of the right-hand side of inequality (17) might diverge. Observe that the convergence of this term

$$\sum_{n=1}^{\infty}\frac{|a_n|^2}{[-2(\mu_n^{+}+\frac{1}{\alpha})]^{\gamma}}\tag{19}$$

depends on the initial conditions, in the sense that if they are regular enough then (19) will converge. More concretely, if the projection of $\vec{U}(0)$ on the subspace generated by $\{\vec{e}_n^+\}$, $\sum_{n=1}^\infty a_n \vec{e}_n^+$, is such that (19) converges, then

$$\|\vec{U}(t)\|_E \leqslant C(\vec{U}(0)) \frac{e^{-\frac{1}{\alpha}t}}{t^{\frac{\gamma}{2}}}$$

where $C(\vec{U}(0))$ stands for a certain constant depending on the initial condition. Intuitively, the appropriate space to choose the initial conditions according to this would be the domain

$$\mathcal{D}\left(\left(-L - \frac{1}{\alpha}\,\mathrm{Id}\right)^{-\frac{\gamma}{2}}\right).$$

So, we need to be more explicit about the coordinates of this projection, especially in terms of each component of $\vec{U}(0) = (u(0), v(0))^T$. In this sense, it is easier to work with initial conditions of the form:

$$\vec{U}(0) = \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^\infty \alpha_n \varphi_n \\ \sum_{n=1}^\infty \beta_n \varphi_n \end{pmatrix} = \sum_{n=1}^\infty \alpha_n \begin{pmatrix} \varphi_n \\ 0 \end{pmatrix} + \beta_n \begin{pmatrix} 0 \\ \varphi_n \end{pmatrix}. \tag{20}$$

Remember that $\{\varphi_n\}$ stands for the basis of $\mathcal{H}$ consisting of orthonormal eigenfunctions of the operator A (see Proposition 10). It is easy to see that

$$\left\{ \begin{pmatrix} \varphi_n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi_n \end{pmatrix} \right\}$$

is a basis of X (that does not consist of orthonormal eigenfunctions of L). So, in (20) the initial condition $\vec{U}(0)$ is given in terms of α_n, β_n, its coordinates in this new basis.

Our purpose now is to express the basis $\{\vec{e}_n^\pm\}$ in terms of the new one, so as to express a_n in terms of α_n, β_n. By doing this, the conditions on the convergence of (19) will be expressed in terms of conditions on these new coefficients.

To relate the coefficients in these two bases we observe first that the orthonormal one consists of the following eigenfunctions:

$$\vec{e}_n^\pm = \frac{\sqrt{2}}{\sqrt{\lambda_n^2 \alpha^2 - 4\lambda_n}} \begin{pmatrix} \varphi_n \\ \mu_n^\pm \varphi_n \end{pmatrix}.$$

A simple calculation gives

$$\begin{pmatrix} \varphi_n \\ 0 \end{pmatrix} = -\frac{1}{\sqrt{2}} \mu_n^- \vec{e}_n^+ + \frac{1}{\sqrt{2}} \mu_n^+ \vec{e}_n^-, \qquad \begin{pmatrix} 0 \\ \varphi_n \end{pmatrix} = \frac{1}{\sqrt{2}} \vec{e}_n^+ - \frac{1}{\sqrt{2}} \vec{e}_n^-$$

and

$$\vec{U}(0) = \sum_{n=1}^\infty \left(\frac{-\alpha_n \mu_n^- + \beta_n}{\sqrt{2}} \right) \vec{e}_n^+ + \left(\frac{\alpha_n \mu_n^+ - \beta_n}{\sqrt{2}} \right) \vec{e}_n^-.$$

In particular,

$$a_n = \frac{-\alpha_n \mu_n^- + \beta_n}{\sqrt{2}}, \qquad b_n = \frac{\alpha_n \mu_n^+ - \beta_n}{\sqrt{2}}. \tag{21}$$

Now we want to prove the convergence of (19) where a_n are given by (21). Since $\mu_n^+ = (-\lambda_n + \sqrt{\lambda_n^2\alpha^2 - 4\lambda_n})/2$ and $\lambda_n \to +\infty$ when $n \to \infty$ (see Proposition 10), we have that

$$\frac{1}{[-2(\mu_n^+ + \frac{1}{\alpha})]^\gamma} = \frac{\alpha^{3\gamma}}{2^\gamma}\lambda_n^\gamma\left[1 + \frac{2\gamma}{\alpha^2}\left(\frac{1}{\lambda_n}\right) + O\left(\frac{1}{\lambda_n^2}\right)\right] \leqslant C_3\lambda_n^\gamma$$

for some $C_3 > 0$ independent of n. Then, we can bound (19) as

$$\sum_{n=1}^{\infty}\frac{|a_n|^2}{[-2(\mu_n^+ + \frac{1}{\alpha})]^\gamma} \leqslant C_3\sum_{n=1}^{\infty}|a_n|^2\lambda_n^\gamma. \tag{22}$$

Using now the expression of a_n in terms of α_n, β_n given in (21), we have that (22) is equal to

$$\frac{C_3}{2}\sum_{n=1}^{\infty}\left|-\alpha_n\mu_n^- + \beta_n\right|^2\lambda_n^\gamma. \tag{23}$$

Observe that in the previous expression we can still expand μ_n^- as

$$\mu_n^- = \lambda_n\left(-\alpha + O\left(\frac{1}{\lambda_n}\right)\right) \leqslant -\alpha\lambda_n + C_4.$$

Using this expansion, we have that

$$\begin{aligned}(23) &\leqslant \frac{C_3}{2}\sum_{n=1}^{\infty}\left(|\alpha\lambda_n\alpha_n + \beta_n|\lambda_n^{\gamma/2} + C_4|\alpha_n|\lambda_n^{\gamma/2}\right)^2 \\ &\leqslant C_3\sum_{n=1}^{\infty}\left(|\alpha\lambda_n\alpha_n + \beta_n|\lambda_n^{\gamma/2}\right)^2 + \left(C_4|\alpha_n|\lambda_n^{\gamma/2}\right)^2.\end{aligned}$$

For the convergence of the previous expression it is sufficient to choose an initial condition $\vec{U}(0) = (u(0), v(0))^T \in \mathcal{D}(A) \times \mathcal{H}$ such that

$$u(0) \in \mathcal{D}\left(A^{\gamma/2}\right) \quad \text{and} \quad \alpha Au(0) + v(0) \in \mathcal{D}\left(A^{\gamma/2}\right).$$

With these conditions, we have explicitly bounded (19) as

$$\sum_{n=1}^{\infty}\frac{|a_n|^2}{[-2(\mu_n^+ + \frac{1}{\alpha})]^\gamma} \leqslant C_5\left[\sum_{n=1}^{\infty}\left(|\alpha\lambda_n\alpha_n + \beta_n|\lambda_n^{\gamma/2}\right)^2 + \left(|\alpha_n|\lambda_n^{\gamma/2}\right)^2\right] \tag{24}$$

with $C_5 \geqslant 0$ being independent of the initial conditions. Hence, the expression in (17) converges. Using (18) and (24) in (17), we can conclude that if $\vec{U}(t)$ is the solution whose initial condition is given by $\vec{U}(0) = (u(0), v(0))^T$, satisfying that $u(0)$ and $\alpha Au(0) + v(0)$ belong to $\mathcal{D}(A^{\gamma/2})$, then

$$\left\|\vec{U}(t)\right\|_E \leqslant K\left(\|u(0)\|_{\mathcal{D}(A)} + \|v(0)\|_{\mathcal{H}} + \|u(0)\|_{\mathcal{D}(A^{\frac{\gamma}{2}})} + \|\alpha Au(0) + v(0)\|_{\mathcal{D}(A^{\frac{\gamma}{2}})}\right)\frac{e^{-\frac{1}{\alpha}t}}{t^{\frac{\gamma}{2}}}.$$

In other words,

$$\left\|\vec{U}(t)\right\|_E \leqslant K\left\|\vec{U}(0)\right\|_{R_\gamma}\frac{e^{-\frac{1}{\alpha}t}}{t^{\frac{\gamma}{2}}}$$

for a certain $K \geqslant 0$ independent of the initial conditions. This completes the proof of Theorem 6. □

Remark 11. Observe that conditions

$$\vec{U}(0) \in R_\gamma \quad \text{and} \quad \vec{U}(0) \in \mathcal{D}\left(\left(-L - \frac{1}{\alpha}\,\mathrm{Id}\right)^{-\frac{\gamma}{2}}\right)$$

are apparently not the same. Nevertheless it can be seen that they are equivalent. To see this, we only have to consider the following operators Q and Z in X

$$Q = \left(-L - \frac{1}{\alpha}\,\mathrm{Id}\right)^{-\frac{\gamma}{2}}, \qquad Z\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A^{\frac{\gamma}{2}-1}(\alpha A u + v) \\ 0 \end{pmatrix}$$

both with domain R_γ. By using expansions in series in λ_n, it can be seen that $Q - Z$ is bounded. This allows us to say that, essentially, both operators are the same.

The result on the optimality of the decay rate of the smoother solutions has been stated in Theorem 7, which we now proceed to prove.

Proof of Theorem 7. We are going to prove this result by contradiction. So, let us suppose that there exists a function $G : R_\gamma \to [0, \infty)$ and a function $\phi : [0, \infty) \to (0, \infty)$ with $\phi(t) \to 0$ when $t \to \infty$ such that

$$\left\|\vec{U}(t)\right\|_X \leqslant G\big(\vec{U}(0)\big)\frac{e^{-\frac{1}{\alpha}t}}{t^{\frac{\gamma}{2}}}\phi(t) \quad \text{for all } \vec{U}(0) \in R_\gamma. \tag{25}$$

This assumption together with Theorem 6 implies that the family of operators

$$\left\{\frac{t^{\frac{\gamma}{2}} e^{\frac{1}{\alpha}t}}{\phi(t)} e^{Lt},\ t \in [0, \infty)\right\} \subset \mathcal{L}(R_\gamma, X)$$

is pointwise bounded. Thus, by the Banach–Steinhaus theorem, these operators are uniformly bounded:

$$t^{\frac{\gamma}{2}} e^{\frac{1}{\alpha}t}\left\|\vec{U}(t)\right\|_X \leqslant K \cdot \left\|\vec{U}(0)\right\|_{R_\gamma}\phi(t) \quad \text{for all } \vec{U}(0) \in R_\gamma, \ \text{for all } t \geqslant 0. \tag{26}$$

However we can find a set of initial conditions $\vec{r}_n(0) \in R_\gamma$ and a sequence of times $t_n \to \infty$ that will contradict (26) and, therefore, (25).

First, let us begin with an initial condition $\vec{r}_n(0) = c_n\binom{\varphi_n}{\mu_n^+\varphi_n}$ such that $\|\vec{r}_n(0)\|_{R_\gamma} = 1$. Only a calculation is required to see that

$$\vec{r}_n(0) = \frac{1}{(\lambda_n + |\mu_n^+|) + \lambda_n^{\frac{\gamma}{2}}(1 + |\alpha\lambda_n + \mu_n^+|)}\begin{pmatrix} \varphi_n \\ \mu_n^+\varphi_n \end{pmatrix}. \tag{27}$$

Observe that if $\vec{r}_n(t)$ is the solution whose initial condition is given by $\vec{r}_n(0)$, then $\vec{r}_n(t) = e^{\mu_n^+ t}\vec{r}_n(0)$. Therefore, its norm is given by

$$\left\|\vec{r}_n(t)\right\|_X^2 = \left(\frac{\lambda_n + |\mu_n^+|}{(\lambda_n + |\mu_n^+|) + \lambda_n^{\frac{\gamma}{2}}(1 + |\alpha\lambda_n + \mu_n^+|)}\right)^2 e^{2\mu_n^+ t} =: d_n e^{2\mu_n^+ t}.$$

Let us play a little bit with this formula to obtain an appropriate sequence $t_n \to \infty$. The previous expression can be written as

$$\begin{aligned}\left\|\vec{r}_n(t)\right\|_X^2 &= d_n e^{2(\mu_n^+ + \frac{1}{\alpha})t} e^{-\frac{2}{\alpha}t} \\ &= d_n\left[-2\left(\mu_n^+ + \frac{1}{\alpha}\right)t\right]^{-\gamma}\left[-2\left(\mu_n^+ + \frac{1}{\alpha}\right)t\right]^{\gamma} e^{2(\mu_n^+ + \frac{1}{\alpha})t} e^{-\frac{2}{\alpha}t}.\end{aligned}$$

Remember that $x^\gamma e^{-x} \leqslant \gamma^\gamma e^{-\gamma}$ for all $x > 0$. And observe that the equality happens for $x = \gamma$. This suggests taking $t_n \geqslant 0$ such that

$$\left[-2\left(\mu_n^+ + \frac{1}{\alpha}\right)t_n\right] = \gamma.$$

That is

$$t_n = \frac{-\gamma}{2(\mu_n^+ + \frac{1}{\alpha})} \to \infty \quad \text{when } n \to \infty. \tag{28}$$

Then, we can write inequality (26) for $\vec{r}_n(0)$ and t_n given in (27) and (28), respectively, to obtain the following inequality:

$$t_n^{\frac{\gamma}{2}} e^{\frac{1}{\alpha}t_n} \frac{\lambda_n + |\mu_n^+|}{(\lambda_n + |\mu_n^+|) + \lambda_n^{\frac{\gamma}{2}}(1 + |\alpha\lambda_n + \mu_n^+|)} e^{-\frac{\gamma}{2}} e^{-\frac{1}{\alpha}t_n} \leqslant K \cdot \phi(t_n).$$

It can be written as

$$\left(\frac{-\gamma}{2(\mu_n^+ + \frac{1}{\alpha})}\right)^{\frac{\gamma}{2}} \frac{\lambda_n + |\mu_n^+|}{(\lambda_n + |\mu_n^+|) + \lambda_n^{\frac{\gamma}{2}}(1 + |\alpha\lambda_n + \mu_n^+|)} e^{-\frac{\gamma}{2}} \leqslant K \cdot \phi(t_n). \tag{29}$$

Observe that, as $n \to \infty$, the right-hand side of (29) tends to 0 (because $t_n \to \infty$). Let us find the limit of the left-hand side. By expanding it in powers of λ_n, we can see that

$$\left(\frac{-\gamma}{2(\mu_n^+ + \frac{1}{\alpha})}\right)^{\frac{\gamma}{2}} \frac{\lambda_n + |\mu_n^+|}{(\lambda_n + |\mu_n^+|) + \lambda_n^{\frac{\gamma}{2}}(1 + |\alpha\lambda_n + \mu_n^+|)} = \frac{(\frac{\gamma\alpha^3}{2})^{\frac{\gamma}{2}}}{\alpha} + O\left(\frac{1}{\lambda_n}\right).$$

Therefore, taking the limit when $n \to \infty$, the expression (29) implies that

$$\frac{(\frac{\gamma\alpha^3}{2})^{\frac{\gamma}{2}}}{\alpha} \leqslant 0.$$

This is obviously impossible when $\alpha, \gamma > 0$ and, hence, assumption (25) cannot be true. An analogous argument applies when $\gamma = 0$, also yielding a contradiction. This completes the proof of this theorem. □

References

[1] A. Bátkai, K.-J. Engel, J. Prüss, R. Schnaubelt, Polynomial stability of operator semigroups, Math. Nachr. 279 (13–14) (2006) 1425–1440.

[2] S. Chen, R. Triggiani, Proof of extensions of two conjectures on structural damping for elastic systems, Pacific J. Math. 136 (1) (1989) 15–55.

[3] P. Freitas, Eigenvalue problems for the wave equation with strong damping, Proc. Roy. Soc. Edinburgh Sect. A 127 (4) (1997) 755–771.

[4] D.H. Griffel, Applied Functional Analysis, Dover Publications, 2002.

[5] T. Kato, Perturbation Theory for Linear Operator, Classics Math., Springer-Verlag, 1980.

[6] G. Lebeau, E. Zuazua, Decay rates for the three-dimensional linear system of thermoelasticity, Arch. Ration. Mech. Anal. 148 (3) (1999) 179–231.
[7] W. Littman, B. Liu, On the spectral properties and stabilization of acoustic flow, SIAM J. Appl. Math. 59 (1) (1999) 17–34.
[8] Z. Liu, B. Rao, Characterization of polynomial decay rate for the solution of linear evolution equation, Z. Angew. Math. Phys. 56 (4) (2005) 630–644.
[9] P. Massatt, Limiting behaviour for strongly damped nonlinear wave equations, J. Differential Equations 48 (1983) 334–349.
[10] M. Pellicer, Large time dynamics of a nonlinear spring-mass-damper model, Nonlinear Anal. 69 (9) (2008) 3110–3127.
[11] M. Pellicer, J. Solà-Morales, Analysis of a viscoelastic spring-mass model, J. Math. Anal. Appl. 294 (2004) 687–698.
[12] M. Pellicer, J. Solà-Morales, Spectral analysis and limit behaviours in a spring-mass system, Commun. Pure Appl. Anal. 7 (3) (2008) 563–577.
[13] K.D. Phung, Polynomial decay rate for the dissipative wave equation, J. Differential Equations 240 (2007) 92–124.
[14] J.E. Muñoz Rivera, R. Quintanilla, On the time polynomial decay in elastic solids with voids, J. Math. Anal. Appl. 338 (2) (2008) 1296–1309.
[15] J.E. Muñoz Rivera, R. Racke, Polynomial stability in two-dimensional magneto-elasticity, IMA J. Appl. Math. 66 (3) (2001) 269–283.
[16] B. Rao, A. Wehbe, Polynomial energy decay rate and strong stability of Kirchhoff plates with non-compact resolvent, J. Evol. Equ. 5 (2) (2005) 137–152.
[17] J. Rauch, X. Zhang, E. Zuazua, Polynomial decay for a hyperbolic–parabolic coupled system, J. Math. Pures Appl. (9) 84 (4) (2005) 407–470.
[18] D.L. Russell, Decay rates for weakly damped systems in Hilbert space obtained with control-theoretic methods, J. Differential Equations 19 (2) (1975) 344–370.
[19] M.A. Shubov, Basis property of eigenfunctions of nonselfadjoint operator pencils generated by the equation of nonhomogeneous damped string, Integral Equations Operator Theory 25 (3) (1996) 289–328.
[20] M.A. Shubov, Spectral operators generated by damped hyperbolic equations, Integral Equations Operator Theory 28 (3) (1997) 358–372.
[21] X. Zhang, E. Zuazua, Unique continuation for the linearized Benjamin–Bona–Mahony equation with space-dependent potential, Math. Ann. 325 (3) (2003) 543–582.
[22] X. Zhang, E. Zuazua, Polynomial decay and control of a 1-d model for fluid–structure interaction, C. R. Math. Acad. Sci. Paris 336 (9) (2003) 745–750.
[23] X. Zhang, E. Zuazua, Control, observation and polynomial decay for a coupled heat–wave system, C. R. Math. Acad. Sci. Paris 336 (10) (2003) 823–828.