J Dyn Diff Equat (2010) 22:399–411
DOI 10.1007/s10884-010-9184-z

Generalized Convergence and Uniform Bounds for Semigroups of Restrictions of Nonselfadjoint Operators

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Received: 11 March 2009 / Published online: 10 August 2010

Abstract In Pellicer (Nonlinear Anal. 69:3110–3127, 2008) we showed the existence of exponentially attracting invariant manifolds for a nonlinear strongly damped viscoelastic problem. In that paper we used an special result, that we did not prove there, concerning certain uniform bounds for families of semigroups generated by restrictions of nonselfadjoint linear operators. The fact of applying this to nonselfadjoint operators represented the main novelty and difficulty of Pellicer (2008). In the present paper we prove this special result that was used in Pellicer (2008). This is done using the notion of generalized convergence between operators.

Keywords Uniform bounds · Nonselfadjoint linear operators · Linear semigroups · Generalized convergence of operators

Mathematics Subject Classification (2000) Primary: 47D06 · 47F05 · Secondary: 35B05 · 35B42 · 47A10

1 Introduction

In our previous work [6] we studied the large time behaviour of the following nonlinear strongly damped wave equation with a dynamic boundary condition:

This paper is dedicated to Professor Jack Hale on the occasion of his 80th birthday.

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$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) - \alpha u_{txx}(x,t) + \varepsilon f\left(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}\right) = 0\,, \quad 0 < x < 1 \\ u(0,t) = 0 \\ u_{tt}(1,t) = -\varepsilon\,[u_x + \alpha u_{tx} + r u_t]\,(1,t) - \varepsilon f\left(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}\right). \end{cases} \tag{1.1}$$

In this problem, $u(x,t)$ represents the longitudinal displacement at time t of the x particle of a viscoelastic spring-mass-damper system where an external force (control) is acting through one of its ends (see [6] for the details on the modelling). As we said, in that previous paper we studied the large time asymptotics of (1.1) when $\varepsilon \to 0$. More specifically, we proved that, under the appropriate hypotheses on the nonlinearity, the PDE (1.1) tends to the nonlinear second order ODE

$$U'' + U + f(U, U') = 0$$

where $U = U(t)$. The main tool used in that work were the so-called exponentially attracting invariant manifolds, an approach which began in works such as [4]. These manifolds are a good way of representing the main elements in the dynamics of a system in terms of an ODE. In the paper [6] our point of view was based in the works of [1] and [2], where some sufficient conditions for the existence of exponentially attracting invariant manifolds for families of operators are given.

In [6] we showed that this approach can be used in (1.1), which is a nonselfadjoint problem. This is, to our opinion, the main novelty (and also difficulty) of that work: applying the point of view of works such as [1] and [2] to a family of nonselfadjoint problems. In showing the existence of exponentially attracting invariant manifolds for a family of operators, the crucial and difficult point is obtaining a uniform bound for certain restrictions of the linear operators (see Lemma 1.2 of [1] or Theorem 4.2 of [6]). When the operators are selfadjoint, this uniform bound can be computed using that the restriction of a selfadjoint operator is also selfadjoint. However, when the linear operators are nonselfadjoint other results must be developed. These results are given, but not proved, in the Appendix of [6]. Hence the main motivation of the present paper: to present a complete proof of these results, which here are given as Theorem 1.

To prove this result we use the generalized convergence between operators, which essentially represents the convergence between their graphs, in a certain distance. This is our particular approach, although it may not be the only one. But it results to be a very appropriate notion of convergence between operators if one wants to compare the corresponding resolvent operators, as we will need. This concept of convergence can be found in the literature (see [5]). Nevertheless, in the present Introduction and also in Sect. 3 below we include and develop the results on this convergence that we use to prove our result.

We now proceed to state the main result of the present paper. In order to do this, we first need to introduce some notation and recall some definitions.

The objective of the present paper is to obtain uniform bounds for a family of restrictions of linear operators converging in the so-called generalized sense. The precise definition of this generalized convergence can be found in Sects. IV.2.1 and IV.2.4 of [5], but we include it here for the self-containedness of the present work.

Let M, N be closed linear manifolds of a Banach space Z (with norm $\|\ \|$). Denote by S_M the unit ball of M, that is

$$S_M = \{u \in M, \|u\| = 1\}.$$

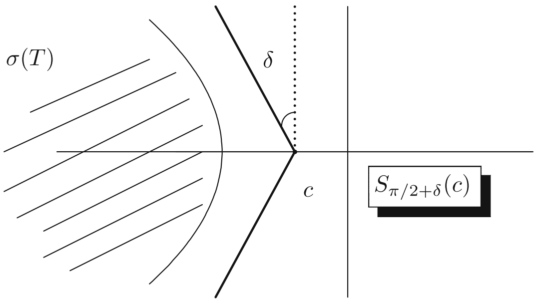


Fig. 1 Sector centered in c and angle δ

We can define

$$\delta(M,N) := \sup_{u\in S_M} \operatorname{dist}(u,N) = \sup_{u\in S_M}\left(\inf_{v\in N}\|u-v\|\right)$$

and

$$\hat{\delta}(M,N) := \max\{\delta\,(M,N)\,,\delta\,(N,M)\}$$

which will be called the **gap** between M and N. Observe that $\hat{\delta}$ is not a distance, but only a semi-distance, as it does not fulfill the triangle inequality (see Sects. IV.2.1 of [5] for more details).

Let now $\mathcal{C}(X,Y)$ be the set of all closed linear operators from the Banach space X into the Banach space Y. For any operator $T\in\mathcal{C}(X,Y)$, let us denote its graph as $G(T)$. This graph is a closed linear manifold of the product space $X\times Y$. Our aim is to introduce a certain notion of convergence for linear operators, using the gap between their graphs.

Definition 1 ([IV.2.4, [5]) Let X, Y be Banach spaces, and let $\{T_n\}_{n\in\mathbb{N}}$ and T be closed linear operators from X to Y. We say that T_n converges to T in the **generalized sense** if $\hat{\delta}(G(T_n),G(T))\to 0$ when $n\to\infty$.

Let us now denote by $S_\delta(c)$ the closed sector centered in $c\in\mathbb{R}$ with angle $\delta\in(0,\pi]$, that is:

$$S_\delta(c) := \{\lambda\in\mathbb{C};\,|\,arg\,(\lambda-c)\,|\le\delta\}.$$

Recall that we say that a closed linear operator T of a Banach space X with dense domain $\mathcal{D}(T)\subset X$ is **sectorial** (with sector centered in $c\in\mathbb{R}$ and angle δ, as in Fig. 1) if there exist $\delta\in(0,\pi/2]$ and $c\in\mathbb{R}$ such that

$$S_{\pi/2+\delta}(c)\setminus\{c\}\ \subset\rho(T)$$

and there exists $C_\delta\ge 1$ such that

$$\|R(\lambda,T)\|\le\frac{C_\delta}{|\lambda-c|}\quad\text{for all }\ c\ne\lambda\in S_{\pi/2+\delta}\,(c) \tag{1.2}$$

From now on let us assume the following hypotheses. Let T_0 be a linear sectorial operator in a Banach space X with all the spectrum in the non-positive half-space except, maybe, $\lambda=0$. That is, with $\mathrm{Re}(\sigma(T_0)\backslash\{0\})<0$. Then, we know that there exists a sector $S_0=S_{\pi/2+\delta_0}(0)$, with $\delta_0\in(0,\pi/2]$, and a constant $C_0\ge 1$ such that

$$S_0\backslash\{0\}\subset\rho(T_0)\quad\text{and}\quad\|R(\lambda,T_0)\|_{\mathcal{L}(X,X)}\le\frac{C_0}{|\lambda|}\ \text{ for all }\lambda\in S_0\backslash\{0\} \tag{1.3}$$

where $\rho(T_0)$ stands for the resolvent set of the operator. Assume also that $\sigma(T_0)$, the spectrum of T_0, is separated into two parts σ_0^1 and σ_0^2 by a constant $c_0 \in \mathbb{R}$ in the following way:

$$\mathrm{Re}(\sigma_0^1) < c_0 < \mathrm{Re}(\sigma_0^2) \leq 0. \tag{1.4}$$

Let $X = X_0 \oplus Y_0$ be the associated decomposition of the whole space X, with $\sigma(T_0|_{X_0}) = \sigma_0^1$ and $\sigma(T_0|_{Y_0}) = \sigma_0^2$ (see [4]).

Let $\{T_\varepsilon\}_{\varepsilon>0}$ be a family of linear sectorial operators in X converging to T_0 in the generalized sense as $\varepsilon \to 0$. Under the appropriate hypotheses, we will prove in Lemma 2 below that the straight line $\{\mathrm{Re}\lambda = c_0\}$ also separates the spectrum of T_ε into two parts. This generates the corresponding decomposition of the whole space X, that is, $X = X_\varepsilon \oplus Y_\varepsilon$, with X_ε and Y_ε being isomorphic to X_0 and Y_0, respectively.

We can now state the main result of the present paper, which is the following theorem, in which a uniform bound for the restrictions of a family of nonselfadjoint operators is obtained.

Theorem 1 (uniform bound on the semigroups generated by restrictions of nonselfadjoint operators) *Let T_0 be a linear sectorial operator in a Banach space X satisfying the hypotheses given in* (1.3) *and* (1.4). *Let $\{T_\varepsilon\}_{\varepsilon>0}$ be a family of linear sectorial operators in X such that:*

(i) *$Re(\sigma(T_\varepsilon)) < 0$ for all $\varepsilon > 0$;*
(ii) *T_ε converges to T_0 in the generalized sense, as $\varepsilon \to 0$;*
(iii) *there exists $\varepsilon_1 > 0$, a constant $C_1 \geq 1$ and a sector $S_1 = S_{\pi/2+\delta_0}(r)$ (with $r > 0$ and the same δ_0 given in* (1.3)*) such that for all $\lambda \in S_1$ and for all $0 < \varepsilon < \varepsilon_1$ we have that:*

$$S_1 \subset \rho(T_\varepsilon) \quad \text{and} \quad \|R(\lambda, T_\varepsilon)\|_{\mathcal{L}(X,X)} \leq \frac{C_1}{|\lambda|}. \tag{1.5}$$

Then, there exists $\varepsilon^* > 0$ such that if $0 < \varepsilon < \varepsilon^*$ the following assertions hold:

1. *There exist an angle $\delta^* \in (0, \delta_0)$ and a constant $C^* \geq 1$, uniform in ε, such that the sector $S = S_{\pi/2+\delta^*}(c_0)$ satisfies that*

$$S\backslash\{c_0\} \subset \rho(T_\varepsilon|_{X_\varepsilon}) \ , \ \|R(\lambda, T_\varepsilon|_{X_\varepsilon})\|_{\mathcal{L}(X_\varepsilon,X_\varepsilon)} \leq \frac{C^*}{|\lambda - c_0|}.$$

for all $\lambda \in S\backslash\{c_0\}$, where c_0 and δ_0 are the constants given in (1.3) *and* (1.4).

2. *Consequently, the following uniform bound on the semigroup generated by the restriction $T_\varepsilon|_{X_\varepsilon}$ holds:*

$$\|e^{T_\varepsilon|_{X_\varepsilon} t}\|_{\mathcal{L}(X_\varepsilon,X_\varepsilon)} \leq M e^{c_0 t}, \quad t \geq 0$$

where $M = M(\delta^*, C^*)$. Therefore, neither M nor c_0 depend on ε when $0 < \varepsilon < \varepsilon^*$.

This theorem and, more specifically, its part 2, is used in [6] for (1.1). Nevertheless, as it can be applied to many other problems we are presenting it in a general form.

As we said, the main contribution of Theorem 1 is that M and c_0 are both uniform in ε for T_ε nonselfadjoint. The difficulties for proving this result are finding a sector and a bound uniform on ε for the resolvent operator of $T_\varepsilon|_{X_\varepsilon}$. This will be done using the ones of $T_0|_{X_0}$ and the generalized convergence theory.

On the other hand, observe that the hypothesis (1.5) is necessary because T_ε being sectorial for each $\varepsilon>0$ only allows us to ensure the existence for each $\varepsilon>0$ of a sector $S_\varepsilon = S_{\pi/2+\delta_\varepsilon}(r_\varepsilon)$, $r_\varepsilon \geq 0$, and a constant $C_\varepsilon \geq 1$ such that $S_\varepsilon \subset \rho(T_\varepsilon)$ and

$$\|R(\lambda, T_\varepsilon)\|_{\mathcal{L}(X,X)} \leq \frac{C_\varepsilon}{|\lambda - r_\varepsilon|} \quad \text{for all } \lambda \in S_\varepsilon\backslash\{r_\varepsilon\}.$$

Also, requiring δ_0 being the same as in (1.3) does not imply a loss of generality. Notice also that requiring (1.5) to be satisfied is not as exceptional as it may seem. For instance, this hypothesis is satisfied when T_ε is a relatively bounded perturbation of T_0, as shown in Lemma A.3 of [6].

The idea behind Theorem 1 is essentially the following. The fact that T_ε converges to T_0 when $\varepsilon \to 0$ in the generalized sense implies that X_ε is isomorphic to X_0 if ε is small enough. Thus, we somehow expect that $T_\varepsilon|_{X_\varepsilon} \to T_0|_{X_0}$ in some sense. If this happened, the sector and the bound for $R(\lambda, T_0|_{X_0})$ could be used for $R(\lambda, T_\varepsilon|_{X_\varepsilon})$ for ε small enough and, therefore, we would have obtained a sector and a bound uniform in ε. The problem is that these resolvent operators are not defined in the same spaces, and therefore it is not so immediate to compare them. In this sense we can say that Theorem 1 somehow generalizes this kind of results from the theory of generalized convergence of operators given in [5].

This paper is organized as follows. In Sect. 2 we present the explicit computation of the bound and exponent of a semigroup generated by a linear sectorial operator. In Sect. 3 we give some technical results using generalized convergence between operators that will be used to prove Theorem 1. The proof of this theorem is done in Sect. 4.

The results presented in this paper are part of the author's PhD thesis ([7]) which was supervised by Prof. J. Solà-Morales. The author would like to thank him for all the fruitful discussions and helpful suggestions.

2 Explicit Bounds for Semigroups of Sectorial Operators

It is well known that given a linear sectorial operator T the following inequality for the corresponding generated semigroup holds:

$$\|e^{Tt}\| \leq Me^{\omega t}$$

for $t \geq 0$ and for certain $M > 0$ and $\omega \in \mathbb{R}$. The question is what do these constants depend on, that is, how to compute them. The answer is simply an adaptation of the results appearing in [3] for sectorial operators with sectors not centered in zero.

Proposition 1 (Adaptation of Proposition 4.3 of Chapter II of [3]) *Let* $(T, \mathcal{D}(T))$ *be a linear sectorial operator with sector* $S_{\pi/2+\delta}(c)$, $c \in \mathbb{R}$ *and* $\delta \in (0, \pi/2]$ *and a constant* $C_\delta \geq 1$ *satisfying* (1.2). *Then, for any* $t \geq 0$ *we have that*

$$\left\| e^{Tt} \right\| \leq M_\delta \, e^{ct}$$

where $M_\delta = M_\delta(C_\delta, \delta)$. *More specifically,*

$$M_\delta = \frac{C_\delta}{2\pi}\left[2\,\frac{e^{-\sin(\delta/2)}}{\sin(\delta/2)} + e\,(\pi + \delta)\right]$$

Proof The proof follows from [3] and can be found in [7]. We adapt the techniques of Proposition 4.3 (chapter II, Sect. 4 of [3]) in order to make explicit what the bound and the exponent depend on. □

So, we can conclude that the constants we are looking for, M_δ and c, only depend on the bound C_δ, the angle δ and the center c of the sector of the resolvent operator. Hence, Theorem 1 will be proved if we can find a sector and a bound uniform in ε for the resolvent operator of $T_\varepsilon|_{X_\varepsilon}$. This is done in Sect. 3 using the generalized convergence of operators, as it allows to compare resolvent operators of operators that converge in the generalized sense.

3 Generalized Convergence and Some Related Results

We devote this section to the concept of generalized convergence of operators and some related technical results that will be used in Sect. 4. In the introduction, we have given the definition of this notion of generalized convergence. Nevertheless, it may be difficult to check it in this way. That is why the following lemma, from Sects. IV.2.6 of [5], is very useful.

Lemma 1 (IV.2.6, [5]) *sLet $\{T_n\}_{n\in\mathbb{N}}$ and T be closed linear operators from X to Y. Then:*

1. *If T^{-1} is well defined and bounded as an operator from Y to X, T_n converges to T in the generalized sense iff for n sufficiently large the operator T_n^{-1} is well defined and bounded and $\|T_n^{-1} - T^{-1}\| \underset{n\to\infty}{\longrightarrow} 0$.*
2. *If T_n converges to T in the generalized sense and B is a linear bounded operator, then $T_n + B \underset{n\to\infty}{\longrightarrow} T + B$ in the generalized sense.*

In this lemma we see that this notion of generalized convergence can be thought as a generalization of the convergence in norm for linear operators that may be unbounded. Also, this concept of generalized convergence is an appropriate way of comparing operators if we are interested in comparing their resolvents, their spectra and corresponding eigenprojections.

This lemma will be applied in the proof of Lemma 4 below.

Lemma 2 *Let T_0 and the family of linear operators $\{T_\varepsilon\}_{\varepsilon>0}$ satisfy the hypotheses of Theorem 1. Then,*

1. *there exists a constant $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the spectrum of each T_ε is separated into two parts, σ_ε^1 and σ_ε^2, by the same straight line $\{Re\,\lambda = c_0\}$. And the associated decomposition of the whole space is given by $X = X_\varepsilon \oplus Y_\varepsilon$, with X_ε and Y_ε being isomorphic to X_0 and Y_0, respectively.*
2. *the corresponding projections onto the spaces X_0 and X_ε converge in norm as follows:*

$$\|P_{X_\varepsilon} - P_{X_0}\|_{\mathcal{L}(X,X)}\,,\ \|P_{Y_\varepsilon} - P_{Y_0}\|_{\mathcal{L}(X,X)} \to 0 \quad \textit{when } \varepsilon \to 0.$$

Proof By Theorem 3.16 of Sects. IV.3.4 of [5], we know that if σ_0^1 and σ_0^2 are separated by a rectifiable, simple and closed curve $\Gamma \subset \rho(T_0)$ (or a finite number of such curves) then, by generalized convergence, the same curve Γ will lie in $\rho(T_\varepsilon)$ and will separate σ_ε^1 and σ_ε^2 in the same way, for ε small enough. So, to prove part 1 we first have to find the appropriate closed curve Γ and then show that the line $\{\mathrm{Re}\,\lambda = c_0\}$ also separates the spectrum $\sigma(T_\varepsilon)$ of T_ε in the same way.

Apart from the sectors S_0 and S_1 given in the hypotheses of Theorem 1, observe that we can guarantee the existence of a third sector. Indeed, as $T_0|_{X_0}$ is sectorial and $\mathrm{Re}(\sigma(T_0|_{X_0})) < c_0$, there exists a sector $S_2 = S_{\pi/2+\delta_2}(c_0)$, with angle $\delta_2 \in (0, \pi/2]$, and a constant $C_2 \geq 1$ such that for all $\lambda \in S_2\backslash\{c_0\}$ we have:

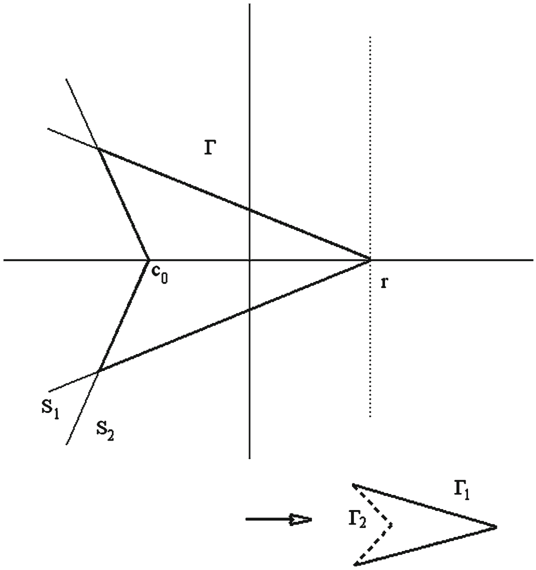
$$S_2\backslash\{c_0\} \subset \rho(T_0|_{X_0})\,,\ \|R(\lambda, T_0|_{X_0})\|_{\mathcal{L}(X_0,X)} \leq \frac{C_2}{|\lambda - c_0|}. \tag{3.1}$$

Without loss of generality, we can suppose that $\delta_2 < \delta_0$. Therefore the boundary of S_2 intersects the boundary of S_1 (see Fig. 2).

Consider now the curve $\Gamma = \partial((\mathbb{C}\backslash\mathrm{int}\,S_1) \cap S_2)$, given in Fig. 2. This curve consists of two parts, $\Gamma_1 = \Gamma \cap \partial S_1$ and $\Gamma_2 = \Gamma \cap \partial S_2$.

The first thing is to observe that $\Gamma \subset \rho(T_0)$. This is obvious for Γ_1 as $\Gamma_1 \subset S_1 \subset S_0\backslash\{0\} \subset \rho(T_0)$ (see (1.3)). For Γ_2, we know by definition that it belongs to ∂S_2, so it is included in

Fig. 2 The curve $\Gamma = \Gamma_1 \cup \Gamma_2$



$\rho(T_0|_{X_0})$ (by (3.1) and also by the fact that $c_0 \in \rho(T_0)$). Also, as $\mathrm{Re}\,\lambda \le c_0$ for all $\lambda \in \Gamma_2$ and by (1.4), we have that $\Gamma_2 \subset \rho(T_0|_{Y_0})$. Thus, $\Gamma_2 \subset \rho(T_0|_{X_0}) \cap \rho(T_0|_{Y_0}) = \rho(T_0)$. And, therefore, $\Gamma \subset \rho(T_0)$. Two additional observations should be made:

1. $\sigma_0^1 = \sigma(T_0|_{X_0}) \subset \mathbb{C}\backslash S_2$. In particular, $\sigma_0^1 \subset \mathrm{ext}\,\Gamma = \mathbb{C}\backslash\overline{\mathrm{int}\,(\Gamma)}$.
2. $\sigma_0^2 = \sigma(T_0|_{Y_0}) \subset \mathbb{C}\backslash S_1$ and $\mathrm{Re}(\sigma(T_0|_{Y_0})) > c_0$. In particular, $\sigma_0^2 \subset \mathrm{int}\,\Gamma$.

So, the curve $\Gamma \subset (\rho(T_0))$ separates the spectrum of T_0 into σ_0^1 and σ_0^2. As $T_\varepsilon \to T_0$ in the generalized sense when $\varepsilon \to 0$ and following Sect. IV.3.4 of [5], we claim that there exists $\varepsilon_0 = \varepsilon_0(\Gamma, T_0) > 0$ such that each $\sigma(T_\varepsilon)$ is separated in the same way as $\sigma(T_0)$ if $0 < \varepsilon < \varepsilon_0$. More specifically, if $0 < \varepsilon < \varepsilon_0$ two additional conclusions can be drawn:

1. $\sigma_\varepsilon^1 := \sigma(T_\varepsilon|_{X_\varepsilon})$, the spectrum of $T_\varepsilon|_{X_\varepsilon}$, satisfies that $\sigma_\varepsilon^1 \subset \mathrm{ext}\,\Gamma$. As we also have $S_1 \subset \rho(T_\varepsilon)$ (see (1.5)), we can ensure that $\mathrm{Re}(\sigma_\varepsilon^1) < c_0$.
2. $\sigma_\varepsilon^2 := \sigma(T_\varepsilon|_{Y_\varepsilon})$, the spectrum of $T_\varepsilon|_{Y_\varepsilon}$, satisfies that $\sigma_\varepsilon^2 \subset \mathrm{int}\,\Gamma$ and $\mathrm{Re}(\sigma_\varepsilon^2) > c_0$. To be precise, the fact that $\mathrm{Re}(\sigma_\varepsilon^2) > c_0$ is not an immediate consequence of $\sigma_\varepsilon^2 \subset \mathrm{int}\,\Gamma$. But it can be proved by considering the separation of the spectrum given by the curve $\Gamma' = \overline{\mathrm{int}\,\Gamma} \cap \{\mathrm{Re}\,\lambda = c_0\}$ and by applying an argument analogous to the one used to see that Γ separates the spectrum of T_0 into σ_0^1 and σ_0^2 and, hence, the spectrum of T_ε.

So, we have proved part 1 of this lemma. Now, part 2 is just an immediate consequence of the generalized convergence between the operators (see Theorem 3.16 of Sect. IV.3.4 of [5]). □

We have seen in Sect. 2 that our purpose is to obtain bounds and sectors for the resolvent operator of $T_\varepsilon|_{X_\varepsilon}$ that are uniform in ε if ε is small enough. From this previous lemma it is intuitively clear that the operators $T_\varepsilon|_{X_\varepsilon}$ and $T_0|_{X_0}$ must be close in some sense. So we expect that the bounds and sector for the resolvent operator of $T_0|_{X_0}$ will be of some use. Actually, the situation is the following:

1. We would like to compare $R(\lambda, T_\varepsilon|_{X_\varepsilon})$ with $R(\lambda, T_0|_{X_0})$, but they are not defined in the same spaces. So, this comparison must be carefully done. We will do it in Lemma 4 using generalized convergence of operators.

2. The generalized convergence between operators allows us to compare the corresponding resolvent operators uniformly inside compact sets (we will see this in Lemma 5 and Proposition 2).
3. Then, we will have uniform constants and sectors for $R(T_\varepsilon|_{X_\varepsilon})$ inside a certain compact set. Outside it, these uniform constants and sectors will be obtained using inequality (1.5).

In the proof of the Theorem 1 the previous strategy will become clearer. Just as a comment, let us say that the comparison between resolvent operators could perhaps also be done using integral formulations of resolvent operators. Nevertheless, this will not be our approach.

Before developing the previous scheme, the following lemma is required.

Lemma 3 *Let $P_{X_\varepsilon}|_{X_0} : X_o \to X_\varepsilon$ be the restriction to X_0 of the projection P_{X_ε} onto X_ε. For ε small enough the following assertions are true:*

1. *This operator has a well-defined inverse, $(P_{X_\varepsilon}|_{X_0})^{-1}$.*
2. *The following convergence property holds:*

$$\left\| P_{X_\varepsilon}|_{X_0} - Id|_{X_0} \right\|_{\mathcal{L}(X_0,X)} \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \textit{and} \quad \left\| \left(P_{X_\varepsilon}|_{X_0}\right)^{-1} - Id|_{X_\varepsilon} \right\|_{\mathcal{L}(X_\varepsilon,X)} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

3. *Both $P_{X_\varepsilon}|_{X_0}$ and $(P_{X_\varepsilon}|_{X_0})^{-1}$ are uniformly bounded in ε (this uniform bound can be taken to be equal to 2, for instance).*
4. *Also, $\rho((P_{X_\varepsilon}|_{X_0})^{-1}\, T_\varepsilon\, (P_{X_\varepsilon}|_{X_0})) = \rho(T_\varepsilon|_{X_\varepsilon})$ and for $\mu \in \rho(T_\varepsilon|_{X_\varepsilon})$ we have that*

$$R(\mu, (P_{X_\varepsilon}|_{X_0})^{-1}\, T_\varepsilon\, (P_{X_\varepsilon}|_{X_0})) = (P_{X_\varepsilon}|_{X_0})^{-1} R(\mu, T_\varepsilon|_{X_\varepsilon})\, (P_{X_\varepsilon}|_{X_0}).$$

Proof To prove part 1, let us begin with the injectivity of $P_{X_\varepsilon}|_{X_0}$. For $u_0 \in X_0$ and just adding and subtracting terms, one can immediately obtain

$$\|P_{X_\varepsilon}|_{X_0} u_0\| = \|P_{X_\varepsilon} u_0\| \geq \left(1 - \|P_{X_0} - P_{X_\varepsilon}\|_{\mathcal{L}(X,X)}\right) \|u_0\|$$

As P_{X_ε} tends to P_{X_0} in norm (see part 2 of Lemma 2), if ε is sufficiently small it is true that $\|P_{X_0} - P_{X_\varepsilon}\|_{\mathcal{L}(X,X)} < 1$ and therefore we can write:

$$\|u_0\| \leq \frac{1}{1 - \|P_{X_0} - P_{X_\varepsilon}\|_{\mathcal{L}(X,X)}} \|P_{X_\varepsilon}|_{X_0} u_0\| \tag{3.2}$$

for all $u_0 \in X_0$. In particular, $P_{X_\varepsilon}|_{X_0}$ is injective when ε is small enough. In order to prove it is also surjective, we have to see that for a given $v_\varepsilon \in X_\varepsilon$ there exists $u_0 \in X_0$ such that:

$$P_{X_\varepsilon}|_{X_0} u_0 = v_\varepsilon. \tag{3.3}$$

If we take $u_0 = P_{X_0}\, u_\varepsilon$ for $u_\varepsilon \in X_\varepsilon$, (3.3) will be proved by finding $u_\varepsilon \in X_\varepsilon$ such that

$$-P_{X_\varepsilon} P_{X_0}\, u_\varepsilon + P_{X_\varepsilon}\, u_\varepsilon + v_\varepsilon = u_\varepsilon.$$

This fixed point problem has a solution if $P_{X_\varepsilon} - P_{X_\varepsilon} P_{X_0}$ is a contraction, which is true when $\varepsilon > 0$ is sufficiently small. Indeed this is true since $P_{X_\varepsilon} - P_{X_\varepsilon} P_{X_0}$ can be also written as $P_{X_\varepsilon}(P_{X_\varepsilon} - P_{X_0})$. And using again that $\|P_{X_\varepsilon} - P_{X_0}\|_{\mathcal{L}(X,X)} \longrightarrow 0$ when $\varepsilon \to 0$, we have that P_{X_ε} is uniformly bounded in ε when ε is small enough and, therefore, $\|P_{X_\varepsilon}\left(P_{X_\varepsilon} - P_{X_0}\right)\|_{\mathcal{L}(X,X)} < 1$. We can conclude then that $P_{X_\varepsilon}|_{X_0}$ is also surjective and, therefore, has a well defined inverse when ε is sufficiently small.

To prove part 2, we use again that $\left\| P_{X_\varepsilon} - P_{X_0} \right\|_{\mathcal{L}(X,X)} \to 0$. In particular,

$$\lim_{\varepsilon\to 0} \left\| P_{X_\varepsilon}|_{X_0} - P_{X_0}|_{X_0} \right\|_{\mathcal{L}(X_0,X)} = \lim_{\varepsilon\to 0} \left\| P_{X_\varepsilon}|_{X_0} - Id|_{X_0} \right\|_{\mathcal{L}(X_0,X)} = 0.$$

Analogously, the corresponding claim follows for $(P_{X_\varepsilon}|_{X_0})^{-1}$.

Let us now prove part 3 of this lemma. From inequality (3.2), we have that

$$\| \left(P_{X_\varepsilon}|_{X_0}\right)^{-1} \|_{\mathcal{L}(X_\varepsilon,X)} \leq \frac{1}{1-\|P_{X_0}-P_{X_\varepsilon}\|_{\mathcal{L}(X,X)}}.$$

In particular, observe that $\left(P_{X_\varepsilon}|_{X_0}\right)^{-1}$ can be bounded for instance by 2 if ε is small enough. And using part 2 of the present lemma, we can obtain that

$$\lim_{\varepsilon\to 0} \|P_{X_\varepsilon}|_{X_0}\|_{\mathcal{L}(X_0,X)} \leq \|Id|_{X_0}\|_{\mathcal{L}(X_0,X)} + \lim_{\varepsilon\to 0} \|P_{X_\varepsilon}|_{X_0} - Id|_{X_0}\|_{\mathcal{L}(X_0,X)} = 1.$$

In particular, we can say that $\|P_{X_\varepsilon}|_{X_0}\|_{\mathcal{L}(X_0,X)} < 2$ if ε is small enough.

Finally, the proof of part 4 of the present lemma is very easy and left to the reader. □

Let us now develop the scheme described before Lemma 3. The first result is about the convergence of restrictions of linear operators that are close in the generalized sense.

Lemma 4 *Let $P_{X_\varepsilon}|_{X_0} : X_o \to X_\varepsilon$ be the restriction to X_0 of the projection P_{X_ε} onto X_ε. Then, $(P_{X_\varepsilon}|_{X_0})^{-1}\, T_\varepsilon\, (P_{X_\varepsilon}|_{X_0})$ converges to $T_0|_{X_0}$ in the generalized sense when $\varepsilon \to 0$.*

Remark 1 As $T_\varepsilon \to T_0$ in the generalized sense, we know that X_ε and X_0 are close spaces when ε is small enough (as they are isomorphic and the corresponding projections converge in norm). That is why we somehow expect that $T_\varepsilon|_{X_\varepsilon} \to T_0|_{X_0}$ in some sense. The problem is that we cannot directly compare these restricted operators, as they act in different spaces. But observe that, as X_0 and X_ε are isomorphic, we can work with $(P_{X_\varepsilon}|_{X_0})^{-1}\, T_\varepsilon\, (P_{X_\varepsilon}|_{X_0})$ instead of $T_\varepsilon|_{X_\varepsilon}$. This operator is now defined on X_0 and, hence, can now be compared to $T_0|_{X_0}$, to which it converges in a generalized sense.

Proof Consider a closed curve $\Gamma \subset \rho(T_0)$ as, for instance, the one considered in the proof of Lemma 2. By the generalized convergence, we have seen that there exists $\varepsilon_0 = \varepsilon_0(\Gamma, T_0)$ such that $\Gamma \subset \rho(T_\varepsilon)$ if $0 < \varepsilon < \varepsilon_0$ and, therefore, we can fix $\mu \in \rho(T_0)\cap\rho(T_\varepsilon)$. In particular, it is also true that $\mu \in \rho(T_0|_{X_0})$ and that $\mu \in \rho(T_\varepsilon|_{X_\varepsilon})$. So, by the last part of Lemma 3 we have that $\mu \in \rho((P_{X_\varepsilon}|_{X_0})^{-1}\, T_\varepsilon\, (P_{X_\varepsilon}|_{X_0}))$. Using part 1 and part 2 of Lemma 1, we are going to see that $(P_{X_\varepsilon}|_{X_0})^{-1}\, T_\varepsilon\, (P_{X_\varepsilon}|_{X_0}) \to T_0|_{X_0}$ in the generalized sense by proving that

$$\left\| R(\mu, (P_{X_\varepsilon}|_{X_0})^{-1}\, T_\varepsilon\, (P_{X_\varepsilon}|_{X_0})) - R(\mu, T_0|_{X_0}) \right\|_{\mathcal{L}(X_0,X_0)} \underset{\varepsilon\to 0}{\longrightarrow} 0$$

By the last part of Lemma 3 and using that the norms in X and X_0 are the same it is immediate that

$$\begin{aligned} &\left\| R(\mu, (P_{X_\varepsilon}|_{X_0})^{-1}\, T_\varepsilon\, (P_{X_\varepsilon}|_{X_0})) - R(\mu, T_0|_{X_0}) \right\|_{\mathcal{L}(X_0,X_0)} \\ &\quad = \left\| (P_{X_\varepsilon}|_{X_0})^{-1}\, R(\mu, T_\varepsilon)\, (P_{X_\varepsilon}|_{X_0}) - R(\mu, T_0)|_{X_0} \right\|_{\mathcal{L}(X_0,X)}. \end{aligned}$$

Let us denote by A the previous expression, that is

$$A := \left\| (P_{X_\varepsilon}|_{X_0})^{-1}\, R(\mu, T_\varepsilon)\, (P_{X_\varepsilon}|_{X_0}) - R(\mu, T_0)|_{X_0} \right\|_{\mathcal{L}(X_0,X)}.$$

By the triangle inequality, we deduce that

$$\begin{aligned} A &\leq \left\| (P_{X_\varepsilon}|_{X_0})^{-1}\, R(\mu, T_\varepsilon)\, (P_{X_\varepsilon}|_{X_0}) - Id|_{X_\varepsilon} R(\mu, T_\varepsilon) \left(P_{X_\varepsilon}|_{X_0}\right) \right\|_{\mathcal{L}(X_0,X)} \\ &\quad + \left\| R(\mu, T_\varepsilon) P_{X_\varepsilon}|_{X_0} - R(\mu, T_0)|_{X_0} \right\|_{\mathcal{L}(X_0,X)} \\ &\leq \left\| (P_{X_\varepsilon}|_{X_0})^{-1} - Id|_{X_\varepsilon} \right\|_{\mathcal{L}(X_\varepsilon,X)} \|R(\mu, T_\varepsilon)\|_{\mathcal{L}(X,X)} \left\| P_{X_\varepsilon}|_{X_0} \right\|_{\mathcal{L}(X_0,X)} \\ &\quad + \left\| R(\mu, T_\varepsilon) P_{X_\varepsilon}|_{X_0} - R(\mu, T_0)|_{X_0} \right\|_{\mathcal{L}(X_0,X)}. \end{aligned}$$

As $R(\mu, T_\varepsilon)$ is acting on the whole space X we can add and subtract $R(\mu, T_0)$, which is defined in the same space:

$$\begin{aligned} A \le & \left\|(P_{X_\varepsilon}|_{X_0})^{-1} - Id|_{X_\varepsilon}\right\|_{\mathcal{L}(X_\varepsilon,X)} \cdot \|R(\mu,T_\varepsilon) - R(\mu,T_0)\|_{\mathcal{L}(X,X)} \cdot \left\|P_{X_\varepsilon}|_{X_0}\right\|_{\mathcal{L}(X_0,X)} \\ & + \left\|(P_{X_\varepsilon}|_{X_0})^{-1} - Id|_{X_\varepsilon}\right\|_{\mathcal{L}(X_\varepsilon,X)} \cdot \|R(\mu,T_0)\|_{\mathcal{L}(X,X)} \cdot \left\|P_{X_\varepsilon}|_{X_0}\right\|_{\mathcal{L}(X_0,X)} \\ & + \left\|R(\mu,T_\varepsilon)P_{X_\varepsilon}|_{X_0} - R(\mu,T_0)|_{X_0}\right\|_{\mathcal{L}(X_0,X)}. \end{aligned} \tag{3.4}$$

Now we add and subtract $R(\mu, T_0)P_{X_\varepsilon}|_{X_0}$ in the right hand side of inequality (3.4) and get:

$$\begin{aligned} A \le & \left\|(P_{X_\varepsilon}|_{X_0})^{-1} - Id|_{X_\varepsilon}\right\|_{\mathcal{L}(X_\varepsilon,X)} \cdot \|R(\mu,T_\varepsilon) - R(\mu,T_0)\|_{\mathcal{L}(X,X)} \cdot \left\|P_{X_\varepsilon}|_{X_0}\right\|_{\mathcal{L}(X_0,X)} \\ & + \left\|(P_{X_\varepsilon}|_{X_0})^{-1} - Id|_{X_\varepsilon}\right\|_{\mathcal{L}(X_\varepsilon,X)} \cdot \|R(\mu,T_0)\|_{\mathcal{L}(X,X)} \cdot \left\|P_{X_\varepsilon}|_{X_0}\right\|_{\mathcal{L}(X_0,X)} \\ & + \left\|R(\mu,T_\varepsilon)P_{X_\varepsilon}|_{X_0} - R(\mu,T_0)P_{X_\varepsilon}|_{X_0}\right\|_{\mathcal{L}(X_0,X)} \\ & + \left\|R(\mu,T_0)P_{X_\varepsilon}|_{X_0} - R(\mu,T_0)Id|_{X_0}\right\|_{\mathcal{L}(X_0,X)}. \end{aligned}$$

By grouping terms in the previous expression, we have that

$$\begin{aligned} A \le & \left\|(P_{X_\varepsilon}|_{X_0})^{-1} - Id|_{X_\varepsilon}\right\|_{\mathcal{L}(X_\varepsilon,X)} \cdot \|R(\mu,T_\varepsilon) - R(\mu,T_0)\|_{\mathcal{L}(X,X)} \cdot \left\|P_{X_\varepsilon}|_{X_0}\right\|_{\mathcal{L}(X_0,X)} \\ & + \left\|(P_{X_\varepsilon}|_{X_0})^{-1} - Id|_{X_\varepsilon}\right\|_{\mathcal{L}(X_\varepsilon,X)} \cdot \|R(\mu,T_0)\|_{\mathcal{L}(X,X)} \cdot \left\|P_{X_\varepsilon}|_{X_0}\right\|_{\mathcal{L}(X_0,X)} \\ & + \|R(\mu,T_\varepsilon) - R(\mu,T_0)\|_{\mathcal{L}(X,X)} \left\|P_{X_\varepsilon}|_{X_0}\right\|_{\mathcal{L}(X_0,X)} + \\ & + \|R(\mu,T_0)\|_{\mathcal{L}(X,X)} \cdot \left\|P_{X_\varepsilon}|_{X_0} - Id|_{X_0}\right\|_{\mathcal{L}(X_0,X)}. \end{aligned} \tag{3.5}$$

Observe that it is true that

$$\|P_{X_\varepsilon}|_{X_0}\|_{\mathcal{L}(X_0,X)} \le 2 \text{ if } 0 < \varepsilon < \varepsilon_2 \quad \text{and} \quad \|R(\mu,T_0)\|_{\mathcal{L}(X,X)} \le N_0$$

(for a certain $\varepsilon_2 > 0$ small enough). These bounds are true due to Lemma 3 (the first bound) and to the fact that $\mu \in \rho(T_0)$ (the second one). Concerning the other terms (the ones with differences between operators inside), we are going to see that, for different reasons, all of them tend to zero when $\varepsilon \to 0$. Indeed, by Lemma 3 we have that:

$$\|(P_{X_\varepsilon}|_{X_0})^{-1} - Id|_{X_\varepsilon}\|_{\mathcal{L}(X_\varepsilon,X)} \to 0\,, \; \|P_{X_\varepsilon}|_{X_0} - Id|_{X_0}\|_{\mathcal{L}(X_0,X)} \to 0 \text{ when } \varepsilon \to 0.$$

And as $T_\varepsilon \to T_0$ in the generalized sense when $\varepsilon \to 0$, by Lemma 1 we have that

$$\|R(\mu,T_\varepsilon) - R(\mu,T_0)\|_{\mathcal{L}(X,X)} \to 0 \text{ when } \varepsilon \to 0.$$

Thus, we conclude that (3.5)$\to 0$ when $\varepsilon \to 0$ and, therefore,

$$\lim_{\varepsilon\to 0} \|R(\mu,(P_{X_\varepsilon}|_{X_0})^{-1}\, T_\varepsilon\,(P_{X_\varepsilon}|_{X_0})) - R(\mu,T_0|_{X_0})\|_{\mathcal{L}(X_0,X_0)} = 0.$$

So, we can say that $(P_{X_\varepsilon}|_{X_0})^{-1}\, T_\varepsilon\,(P_{X_\varepsilon}|_{X_0})$ tends to $T_0|_{X_0}$ in the generalized sense when $\varepsilon \to 0$. □

The second result is adapted from Theorem 3.15 of Sect. IV.3.3 of [5] and gives uniformity of convergence of resolvent operators inside compact sets.

Lemma 5 *Let $K \subset \rho(T_0)$ be a compact set. Then, for a given $\eta > 0$ there exists $\varepsilon_3 = \varepsilon_3(\eta, K, T_0) > 0$ such that if $0 < \varepsilon < \varepsilon_3$ then:*

$$K \subset \rho(T_\varepsilon) \text{ and } \|R(\lambda, T_\varepsilon) - R(\lambda, T_0)\| < \eta \quad \text{for all } \lambda \in K.$$

Proof As we said, it is an immediate application of Theorem 3.15 of Sect. IV.3.3 of [5]. □

Using Lemmas 4 and 5 we can now prove the following result on the existence of a uniform bound for $R(\lambda, T_\varepsilon|_{X_\varepsilon})$ on compact sets. It will be used in the proof of Theorem 1.

Proposition 2 *Let $K \subset \rho(T_0|_{X_0})$ be a compact set where $\|R(\lambda, T_0|_{X_0})\| \leq M_K/|\lambda - c|$ for all $\lambda \in K \setminus \{c\}$, for a certain $M_k \geq 1$ and a certain $c \notin int(K)$, $c \in \mathbb{R}$. Then, for a given $\eta > 0$ there exists $\varepsilon_K = \varepsilon_K(\eta, K, T_0|_{X_0}) > 0$ and $C_K \geq 1$ such that*

$$\|R(\lambda, T_\varepsilon|_{X_\varepsilon})\| \leq \frac{C_K}{|\lambda - c|} \quad \text{if } 0 < \varepsilon < \varepsilon_K, \text{ for all } \lambda \in K \setminus \{c\}$$

where $C_K = 4\left(\eta \max_{\lambda \in K} |\lambda - c| + M_K\right)$.

Observe that C_K is independent of $0 < \varepsilon < \varepsilon_K$ and $\lambda \in K \setminus \{c\}$. We therefore have obtained a bound for the resolvent operator of $T_\varepsilon|_{X_\varepsilon}$, which is uniform in ε and holds for any λ in the compact set K, if ε is small enough.

Proof We have seen in Lemma 4 that the fact that T_ε tends to T_0 in the generalized sense implies that $(P_{X_\varepsilon}|_{X_0})^{-1}\, T_\varepsilon\, (P_{X_\varepsilon}|_{X_0}) \to T_0|_{X_0}$ also in the generalized sense, when $\varepsilon \to 0$. As K is a compact set, Lemma 5 ensures that given $\eta > 0$ there exists $\varepsilon_3 = \varepsilon_3(\eta, K, T_0|_{X_0}) > 0$ such that $K \subset \rho((P_{X_\varepsilon}|_{X_0})^{-1}\, T_\varepsilon\, (P_{X_\varepsilon}|_{X_0})) = \rho(T_\varepsilon|_{X_\varepsilon})$ and that

$$\|R(\lambda, (P_{X_\varepsilon}|_{X_0})^{-1}\, T_\varepsilon\, (P_{X_\varepsilon}|_{X_0})) - R(\lambda, T_0|_{X_0})\|_{\mathcal{L}(X_0,X)} < \eta \text{ for all } \lambda \in K \tag{3.6}$$

if $0 < \varepsilon < \varepsilon_3$. Now we only have to play with the inequalities. First, by the last part of Lemma 3 and adding and subtracting terms we have:

$$\begin{aligned}
&\|R(\lambda, T_\varepsilon|_{X_\varepsilon})\|_{\mathcal{L}(X_\varepsilon,X)} \\
&\quad = \|P_{X_\varepsilon}|_{X_0}(P_{X_\varepsilon}|_{X_0})^{-1}\, R(\lambda, T_\varepsilon|_{X_\varepsilon})\, P_{X_\varepsilon}|_{X_0}(P_{X_\varepsilon}|_{X_0})^{-1}\|_{\mathcal{L}(X_\varepsilon,X)} \\
&\quad \leq \|P_{X_\varepsilon}|_{X_0}\|_{\mathcal{L}(X_0,X)} \\
&\qquad \cdot\|R(\lambda, (P_{X_\varepsilon}|_{X_0})^{-1}T_\varepsilon|_{X_\varepsilon} P_{X_\varepsilon}|_{X_0})\|_{\mathcal{L}(X_0,X)} \cdot \|(P_{X_\varepsilon}|_{X_0})^{-1}\|_{\mathcal{L}(X_\varepsilon,X)} \\
&\quad \leq \|P_{X_\varepsilon}|_{X_0}\|_{\mathcal{L}(X_0,X)} \\
&\qquad \cdot\|R(\lambda, (P_{X_\varepsilon}|_{X_0})^{-1}T_\varepsilon|_{X_\varepsilon} P_{X_\varepsilon}|_{X_0}) - R(\lambda, T_0|_{X_0})\|_{\mathcal{L}(X_0,X)} \cdot \|(P_{X_\varepsilon}|_{X_0})^{-1}\|_{\mathcal{L}(X_\varepsilon,X)} \\
&\qquad +\|P_{X_\varepsilon}|_{X_0}\|_{\mathcal{L}(X_0,X)} \cdot \|R(\lambda, T_0|_{X_0})\|_{\mathcal{L}(X_0,X)} \cdot \|(P_{X_\varepsilon}|_{X_0})^{-1}\|_{\mathcal{L}(X_\varepsilon,X)}
\end{aligned}$$

But using (3.6), the bound for the resolvent of $T_0|_{X_0}$ given in the statement of this proposition, and using that, by Lemma 3, the projections are bounded uniformly on ε (say, for instance, by 2 if $0 < \varepsilon < \varepsilon_2$ for a certain $\varepsilon_2 > 0$ small enough), we can conclude that

$$\|R(\lambda, T_\varepsilon|_{X_\varepsilon})\|_{\mathcal{L}(X_\varepsilon, X)} \leq 2\left(\eta + \frac{M_K}{|\lambda - c|}\right) 2 \leq \frac{C_K}{|\lambda - c|}$$

if $0 < \varepsilon < \varepsilon_K = \min\{\varepsilon_2, \varepsilon_3\}$, for all $\lambda \in K$. □

4 Proof of Theorem 1

Now we have the tools needed to prove the main objective of this work, which is Theorem 1.

Proof We are going to prove that the sector and bounds, uniform in ε, that we are looking for are indeed S_2 and C_2, the sector and constant given in (3.1) (see the proof of Lemma 2). To find the bounds of $R(\lambda, T_\varepsilon|_{X_\varepsilon})$ that we are looking for inside this sector, we are going to divide this sector into two parts: a compact set (where we are going to compare $R(\lambda, T_\varepsilon|_{X_\varepsilon})$ with $R(\lambda, T_0|_{X_0})$ using Proposition 2) and the rest (where the bounds will be achieved by making use of inequality (1.5)).

Which compact set are we going to use? Let Γ be the closed curve defined in the proof of Lemma 2 (see also Fig. 2) and let us consider the compact $K = \overline{\text{int}(\Gamma)}$. Remember that $K \subset S_2$.

The first thing is to observe that

$$S_2 \subset \rho(T_\varepsilon|_{X_\varepsilon}) \tag{4.1}$$

if $0 < \varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$, where $\varepsilon_0, \varepsilon_1 > 0$ are defined below. This is true since $S_2 = K \cup (S_2 \backslash K)$ and:

1. $S_2 \backslash K \subset S_1$ (see proof of Lemma 2), and by (1.5) there exists $\varepsilon_1 > 0$ such that $S_1 \subset \rho(T_\varepsilon)$ if $0 < \varepsilon < \varepsilon_1$, which is a subset of $\rho(T_\varepsilon|_{X_\varepsilon})$.
2. $K \subset \rho(T_\varepsilon|_{X_\varepsilon})$ if $0 < \varepsilon < \varepsilon_0 = \varepsilon_0(\Gamma, T_0)$ (see also the proof of Lemma 2).

Let us now fix our attention on K. As it is a subset of S_2, by (3.1) it is true that $K \backslash \{c_0\} \subset \rho(T_0|_{X_0})$ and that

$$\|R(\lambda, T_0|_{X_0})\|_{\mathcal{L}(X_0, X)} \leq \frac{C_2}{|\lambda - c_0|} \quad \text{for all } \lambda \in K \backslash \{c_0\}.$$

But, by Proposition 2, given $\eta > 0$, there exists $0 < \varepsilon_K = \varepsilon_K(\eta, K, T_0|_{X_0})$ and $C_K \geq 1$ such that

$$\|R(\lambda, T_\varepsilon|_{X_\varepsilon})\|_{\mathcal{L}(X_\varepsilon, X_\varepsilon)} \leq \frac{C_K}{|\lambda - c_0|} \quad \text{for all } \lambda \in K \backslash \{c_0\} \tag{4.2}$$

where $C_K = C_K(\eta, c_0, C_2, K)$ does not depend on ε if $0 < \varepsilon < \varepsilon_K$.

Let us turn our attention to $S_2 \backslash K$. As we have seen, $S_2 \backslash K \subset S_1 \subset \rho(T_\varepsilon|_{X_\varepsilon})$. So, hypothesis (1.5) implies that

$$\|R(\lambda, T_\varepsilon|_{X_\varepsilon})\|_{\mathcal{L}(X_\varepsilon, X_\varepsilon)} \leq \|R(\lambda, T_\varepsilon)\|_{\mathcal{L}(X, X)} \leq \frac{C_1}{|\lambda|} \quad \text{for all } \lambda \in S_2 \backslash K \tag{4.3}$$

if $0 < \varepsilon < \varepsilon_1$. But if we denote by R_Γ the radius of the circle with center $\lambda = 0$ and inscribed in $\overline{\text{int}\,\Gamma}$, it is immediate to see that

$$|\lambda - c_0| \leq |\lambda| + |c_0| \leq \left(1 + \frac{|c_0|}{R_\Gamma}\right)|\lambda| \quad \text{for all } \lambda \in S_2 \backslash K.$$

Therefore, (4.3) can be written as

$$\|R(\lambda, T_\varepsilon|_{X_\varepsilon})\|_{\mathcal{L}(X_\varepsilon,X_\varepsilon)} \leq \frac{C_1\left(1+\dfrac{|c_0|}{R_\Gamma}\right)}{|\lambda - c_0|} \text{ for all } \lambda \in S_2\backslash K.$$

By denoting $C_1\left(1+\dfrac{|c_0|}{R_\Gamma}\right)$ again by C_1, we conclude that

$$\|R(\lambda, T_\varepsilon|_{X_\varepsilon})\|_{\mathcal{L}(X_\varepsilon,X_\varepsilon)} \leq \frac{C_1}{|\lambda - c_0|} \quad \text{for all } \lambda \in S_2\backslash K \tag{4.4}$$

if $0 < \varepsilon < \varepsilon_1$, with $C_1 = C_1(\Gamma, c_0)$ not depending on ε.

Finally, taking $\varepsilon^* = \min\{\varepsilon_0, \varepsilon_1, \varepsilon_K\}$, $C^* = \max\{C_1, C_K\}$ (all uniform in ε) and using (4.1), (4.2) and (4.4), we can conclude that the sector $S = S_2$ independent of $0 < \varepsilon < \varepsilon^*$ is such that

$$S\backslash\{c_0\} \subset \rho(T_\varepsilon|_{X_\varepsilon}) \quad \text{and} \quad \|R(\lambda, T_\varepsilon|_{X_\varepsilon})\|_{\mathcal{L}(X_\varepsilon,X_\varepsilon)} \leq \frac{C^*}{|\lambda - c_0|} \text{ for all } \lambda \in S\backslash\{c_0\}$$

if $0 < \varepsilon < \varepsilon^*$. So, part 1 of the theorem has been proved.

As this part of the theorem is satisfied, part 2 is now just an application of Proposition 1. Using it, we directly obtain that

$$\|e^{T_\varepsilon|_{X_\varepsilon} t}\|_{\mathcal{L}(X_\varepsilon,X_\varepsilon)} \leq M\, e^{c_0 t} \text{ if } 0 < \varepsilon < \varepsilon^*$$

with M only depending on the angle δ^* of the sector S and on the previous constant C^*. In particular, it does not depend on ε if $0 < \varepsilon < \varepsilon^*$. □

Acknowledgements Partially supported by MEC, Spain (MTM2005-07660-C02-02) and MICINN, Spain (MTM2008-06349-C03-02).

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