Internal viscosity and limit behavior in a wave equation with strong damping

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Abstract

We describe a strongly damped wave equation with dynamic boundary conditions as a model for a viscoelastic spring-mass system. In this paper, we present the main results related to the effect of inner spring viscosity, specially in terms of long time behavior of the solutions. Finally, as an important tool, we show the characterization of the essential spectrum for the operator corresponding to this model.

The model and the main results.

We are interested in studying the long time behavior of the solutions of the following problem:

$$\begin{cases} u_{tt} - u_{xx} - \alpha\, u_{txx} = 0, \quad 0 < x < 1, \quad t > 0 \\ u(0,t) = 0 \\ u_{tt}(1,t) = -\varepsilon\, [u_x(1,t) + \alpha\, u_{tx}(1,t) + r\, u_t(1,t)]. \end{cases} \tag{1}$$

This is a wave equation with strong damping and dynamic boundary conditions that models a viscoelastic spring fixed at one end and that has a moving and damped mass at the other one (see [5] for the modelling). Roughly speaking, $u(x,t)$ stands for the displacement at time t of the x particle of the spring and several parameters appear in the equation: $\alpha \geq 0$ represents the internal viscosity or damping, $r > 0$ the external damping and $\varepsilon > 0$ is essentially the inverse of the mass. A detailed list of references to related problems can be found also in [5].

Our approach to this problem is the following. Classically, a damped spring-mass system has been modeled by the second order ODE:

$$m\, u''(t) = -k\, u(t) - d\, u'(t) \tag{2}$$

where $k > 0$ is the recovery spring constant and $d \geq 0$ is the damping coefficient. In this equation, the system is treated as a space-homogeneous system and neither the action of internal-external damping or viscosity are separated. That is why in [5] we proposed the PDE model (1) instead of the ODE one. At that point, we are also interested in comparing both approaches to the same system. Basically, our question will be if the classical ODE is the limit of the PDE model in some sense, that is, if the solutions of PDE system tend to the solutions of an ODE of type (2) when $t \to \infty$.

The main tool we use to answer this question are the dominant eigenvalues. A detailed definition is given in [6], but roughly speaking they are finite subsets of isolated eigenvalues with real part strictly greater than the rest of the spectrum. This is why we can say that the long time behavior of the solutions of the PDE will be given by those subsets and, hence, the corresponding ODE can be thought as the limit of the PDE.

In the previous work [5] we proved that for fixed α, $r > 0$ and ε small enough, the partial differential equation model (1) admits two dominant eigenvalues, and therefore it was proved the existence of a second order ODE of the same type as (2) as the limit of our model for these values of the parameters. In a future work a nonlinear version of this limit will also be studied.

However, this may not always occur. In the paper [6], in contrast with [5], they are shown three interesting situations for (1), all of them related with internal viscosity, where either the non-existence of a finite subset of dominant eigenvalues can be proved or where there exists such a finite subset but the resulting ODE is not of the same type as (2). Thus, the existence of a limit ODE of type (2) is not an automatic property for model (1) as in principle one may think, but it only holds in some regions in the space of parameters (ε, α, r). These three cases are summarized in the following statements.

The first of these three cases is $\alpha = 0$, that is the purely elastic spring, with an external damper but without internal viscosity. Although it may seem the most similar situation to the one modeled by the classical equation (2), it is proved the non-existence of a limit ODE in this case.

Theorem 1 *When $\alpha = 0$ we have the following results:*

i) The spectrum of A_α with $\alpha = 0$ consists only of eigenvalues $\{\lambda_n\}$, $n \in \mathbb{N}$, with strictly negative real part that approach the imaginary axis as $|\lambda_n| \to \infty$. Therefore, does not exist a finite subset of dominant eigenvalues.

ii) In this case, it can also be seen that all solutions tend to zero as $t \to \infty$, but there exist solutions which tend to zero with arbitrarily slow rate.

The second case is $\alpha \sim 0$, that is a spring with small internal viscosity. This case is studied as a singular perturbation of the previous one, but exhibits very different properties. The main result is the following.

Theorem 2 *A_α admits a finite subset of maximal dominant eigenvalues for each $\alpha > 0$ if α is small enough. But this set does not depend continuously on α as $\alpha \to 0$. More precisely, neither the number of these eigenvalues nor their positions are continuous on α as $\alpha \to 0$.*

Finally the case with large α represents a spring with large internal damping. The asymptotic dynamics of this situation is also not well approximated by any ODE of type (2) and also a kind of infinite-dimensional overdamping occurs for certain values of ε and r.

Theorem 3 *For some values of ε and r, and for α large enough, the operator A_α does not admit a finite subset of dominant eigenvalues. Actually, all the eigenvalues $\{\lambda_n\}$, $n \in \mathbb{N}$, are real with $-\infty < \lambda_n < -1/\alpha$, with a subsequence λ_n^+ that accumulates at $-1/\alpha$ and the rest of them accumulating at $-\infty$.*

These three results allow us to say that an ordinary differential equation may not be the most appropriate model to describe the dynamics of a viscous spring-mass-damper system, at least in some cases and that these cases turn to have rellevant physical meaning.

The proofs of these three results are the aim of our work [6]. All the needed tools are set there except for one, which appears but it is not proved. It is the essential spectrum of the operator related to problem (1) that, although it is used in showing some of the previous results (and even appears in [5]), is not proved in none of these papers. As we find this result interesting by himself, we use this opportunity to develop this study with more carefulness. This will be done in the third part of this paper but, before that, we will set the appropriate functional framework where this problem is well posed, as it will be needed in studying this essential spectrum.

Functional framework.

Although we consider problem (1) with $\varepsilon > 0$, the functional framework given below also suits also in the case of $\varepsilon = 0$.

Let us write problem (1) in the evolution form

$$\frac{d}{dt}V = A_\alpha V \quad (\alpha \geq 0). \tag{3}$$

As in many problems with dynamic boundary conditions it is appropriate to work in spaces whose elements are pairs of a function and its boundary value. Moreover, since we are writing a second order evolution equation as a first order system our phase spaces will consist of pairs of such pairs. In this point we are strongly influenced by the work of M. Grobbelaar-van Dalsen [3]. The following definitions also appear in previous works, but we summarize them here for a better comprehension of the subsequent results.

We define the following spaces:

$$X_0 = L^2(0,1) \times \mathbb{C},$$

$$X_1 = \{(u, \gamma) \in H^1(0,1) \times \mathbb{C},\ u(0) = 0,\ u(1) = \gamma\} \subset H^1(0,1) \times \mathbb{C},$$

$$X_2 = \{(u,\gamma) \in H^2(0,1)\times\mathbb{C},\ u(0)=0,\ u(1)=\gamma\} \subset H^2(0,1)\times\mathbb{C}$$

and $\mathcal{H} = X_1 \times X_0$, that is a Hilbert space with a the following inner product:

$$\left\langle \begin{pmatrix}(u_1,u_1(1))\\(u_0,\gamma_0)\end{pmatrix}, \begin{pmatrix}(v_1,v_1(1))\\(v_0,\beta_0)\end{pmatrix}\right\rangle_{\mathcal{H}} = \int_0^1 (u_1)_x\,(\overline{v}_1)_x\,dx \ + \ \int_0^1 u_0\,\overline{v}_0\,dx \ + \gamma_0\,\overline{\beta}_0. \tag{4}$$

The square of the norm defined by this scalar product (4) is related with the total physical energy of the system. This norm in $\mathcal{H}$ will be denoted simply by $\|\ \|$.

We define $(A_\alpha, \mathcal{D}(A_\alpha))$ as follows:

$$\mathcal{D}(A_\alpha) = \left\{ \begin{pmatrix}(u,u(1))\\(v,v(1))\end{pmatrix} \in X_1\times X_1,\ (u+\alpha\,v)\in H^2(0,1)\right\} \subset \mathcal{H}$$

is the domain of A_α, which is:

$$A_\alpha \begin{pmatrix}(u,u(1))\\(v,v(1))\end{pmatrix} = \begin{pmatrix}(v,v(1))\\ (\,(u+\alpha\,v)_{xx},\ -\varepsilon\,(u+\alpha\,v)_x(1)\ -\ \varepsilon\,r\,v(1)\,)\end{pmatrix}.$$

Then, equation (1) can be written as the evolution equation:

$$\frac{d}{dt}V = A_\alpha V, \qquad t\in(0,\infty)$$

with $V = \begin{pmatrix}(u,u(1))\\(u_t,u_t(1))\end{pmatrix} \in \mathcal{D}(A_\alpha)$. This is a well-posed problem, as it can be seen in [6].

The essential spectrum.

In this section we will give the characterization of the essential spectrum of A_α when $\alpha>0$ as well as a complete proof of it. In the case $\alpha=0$ the spectrum consists only of eigenvalues (see [6] for the proof). There are different definitions for the essential spectrum. We will use the notion that appears in [4] and that is previously introduced in [2].

Definition 1 (Gohberg, Kreĭn [2]) *Let T be a linear operator in a Banach space X and with domain $\mathcal{D}(T)\subset X$. If we denote by $\sigma(T)$ the spectrum of T, we call $\lambda\in\sigma(T)$ a* ***normal eigenvalue*** *of T if λ is an isolated point of $\sigma(T)$, of finite algebraic multiplicity and $T-\lambda Id$ has closed range in X.*

We say that $\lambda\in\mathbb{C}$ is a ***normal point*** *of T if either λ is in the resolvent set of T ($\lambda\in\rho(T)$) or λ is a normal eigenvalue of T.*

We call the ***essential spectrum of*** *T, and we denote it by $\sigma_{ess}(T)$, the set of points $\lambda\in\sigma(T)$ that are not normal eigenvalues.*

Under this definition, the essential spectrum is invariant under relatively compact perturbations, as the following result from [4] shows.

Theorem 4 (Henry [4]) *Let $T+S$ be a relatively compact perturbation of a closed operator T, that is, S is a linear operator with $\mathcal{D}(S)\supset\mathcal{D}(T)$ and such that $S(T-\lambda_0\,Id)^{-1}$ is compact for some $\lambda_0\in\mathbb{C}$. Let $\mathcal{U}$ be any open connected subset de $\mathbb{C}$ consisting only of normal points of T. Then, either $\mathcal{U}$ consists only of normal points of $T+S$ or it only contains eigenvalues of $T+S$.*

With this definitions and results, we are able to state the main result of this paper.

Theorem 5 *The essential spectrum of the operator A_α when $\alpha\geq 0$ is:*

$$\sigma_{ess}\left(A_\alpha\right) = \left\{\frac{-1}{\alpha}\right\}.$$

Our objective is to prove this theorem, but as we have the result from theorem 4 we can consider to work with an easier perturbed problem. That is why we will concentrate on the following operator:

$$(A_\alpha(0)+B)\begin{pmatrix}(u,u(1))\\(v,v(1))\end{pmatrix} = \begin{pmatrix}(v,v(1))\\(\,(u+\alpha\,v)_{xx},0\,)\end{pmatrix} + \begin{pmatrix}(0,0)\\ \left(-\dfrac{1}{\alpha}\,v,0\right)\end{pmatrix}$$

where $A_\alpha(0)$ is the operator A_α with $\varepsilon = 0$, that has also domain $\mathcal{D}(A_\alpha)$. Observe that this operator $A_\alpha(0) + B$ comes from the also perturbed problem:

$$\begin{cases} \frac{d}{dt}(u_t - \alpha\, u_{xx}) = \frac{-1}{\alpha}\,(u_t - \alpha\, u_{xx}), \quad 0 < x < 1, \; t > 0 \\ u(0,t) = 0, \quad t > 0 \\ u_{tt}(1,t) = 0, \quad t > 0 \end{cases} \tag{5}$$

where we suspect which may be the convenient change of variables to apply. This is the operator of which we will compute the essential spectrum, as follows.

Lemma 6

$$\sigma_{ess}\left(A_\alpha(0) + B\right) \; = \left\{\frac{-1}{\alpha}\right\}.$$

Proof Problem (5) suggests us the following change of variables for $A_\alpha(0) + B$:

$$\begin{pmatrix} (u, u(1)) \\ (v, v(1)) \end{pmatrix} \in \mathcal{D}(A_\alpha) \;\longrightarrow\; \begin{pmatrix} (f, f(1)) \\ (g, b) \end{pmatrix} = \begin{pmatrix} (u, u(1)) \\ (\, v - \alpha\, u_{xx}, v(1)\,) \end{pmatrix}.$$

As the old variables are in $\mathcal{D}(A_\alpha) \subset \mathcal{H}$ the new variables have sense in the following space defined from the old variables domain:

$$\mathrm{E} = \left\{ \begin{pmatrix} (f, f(1)) \\ (g, b) \end{pmatrix}, \; (f, f(1)) \in X_1, \; g \in H^{-1}(0,1), \; f + \alpha(g + \alpha f_{xx}) \in H^2(0,1), \; b = g + \alpha f_{xx}|_{x=1} \right\}$$

with $E \subset X$, where:

$$X = \left\{ \begin{pmatrix} (f, f(1)) \\ (g, b) \end{pmatrix} \in X_1 \times (H^{-1}(0,1) \times \mathbb{C}), \; g + \alpha f_{xx} \in L^2(0,1) \right\}$$

and with $H^{-1}(0,1)$ denoting the dual space of $H_0^1(0,1)$. Also, the corresponding norm can be obtained and is:

$$\left\| \begin{pmatrix} (f, f(1)) \\ (g, b) \end{pmatrix} \right\|_X = \int_0^1 |f_x|^2\, dx + \int_0^1 |g + \alpha f_{xx}|^2\, dx + |b|^2 .$$

Under the previous change of variables, the perturbed operator can be written as the following upper triangular operator:

$$(A_\alpha(0) \; + \; B) \begin{pmatrix} (f, f(1)) \\ (g, b) \end{pmatrix} \; = \; T \begin{pmatrix} f, f(1) \\ g, b \end{pmatrix} \; = \; \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} (f, f(1)) \\ (g, b) \end{pmatrix} \; = \; \begin{pmatrix} (\alpha\, \partial_x^2 f + g, b) \\ (-\frac{1}{\alpha}\, g, 0) \end{pmatrix}$$

where

$$T_1 : X_1 \longrightarrow H^{-1} \times \mathbb{C}\;,\; T_2 : H^{-1} \times \mathbb{C} \longrightarrow H^{-1} \times \mathbb{C} \text{ and } T_3 : H^{-1} \times \mathbb{C} \longrightarrow H^{-1} \times \mathbb{C}.$$

It is may be easier to deal with these operators if we think them as:

$$T_1 = \begin{pmatrix} \alpha\, \partial_x^2 & 0 \\ 0 & 0 \end{pmatrix}\;,\; T_2 = \begin{pmatrix} Id & 0 \\ 0 & 1 \end{pmatrix} \text{ and } T_3 = \begin{pmatrix} -\frac{1}{\alpha}\, Id & 0 \\ 0 & 0 \end{pmatrix}$$

applied to elements of the type $(f, f(1), g, b)'$.

Following the same idea as in [1], we will relate the spectrum and the resolvent set of an operator of the same type as T with those of T_1, T_2 and T_3. In this sense, it can be easily seen that

$$\sigma \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} = \sigma(T_1) \cup \sigma(T_3) = \left\{0, -k\pi^2, k = 1, 2, \ldots\right\} \cup \left\{\frac{-1}{\alpha}, 0\right\}. \tag{6}$$

We now analyze these different possibilities in terms of $\sigma(T)$. Let us begin with $\lambda = \frac{-1}{\alpha}$. In this case, we can check that all the vectors of the form

$$\begin{pmatrix} (s(x), 0) \\ (\,-\frac{1}{\alpha} s(x) - \alpha\, s''(x), 0\,) \end{pmatrix}$$

are eigenvectors of T of eigenvalue $\lambda = \frac{-1}{\alpha}$, for all $s(x)$ in the corresponding domain. So, $\lambda = \frac{-1}{\alpha} \in \sigma(T)$ and has infinite algebraic multiplicity as an eigenvalue of T. That is, $\lambda = \frac{-1}{\alpha}$ belongs to the essential spectrum of $A_\alpha(0) + B$.

Let us now take $\lambda \in \sigma(T) \setminus \left\{\frac{-1}{\alpha}\right\}$. Because of the structure of $\sigma(T)$ seen in (6) and also from the fact that $\sigma(T_1)$ consists only of normal eigenvalues (because T_1 has compact resolvent), we are able to say that λ is also an isolated point of $\sigma(T)$, also with finite algebraic multiplicity as an eigenvalue of T. In order to see whether it is a normal eigenvalue of T, the only thing that still remains to be checked is if $T - \lambda\, Id$ has closed range in X. This has to be done in the case of $\lambda \neq 0$ and $\lambda = 0$ separatedly, but in both cases it can be seen following the classical way of proving that a certain operator has closed range together with the fact that, as λ is a normal eigenvalue of T_1, $T_1 - \lambda\, Id$ has closed range in $H^{-1}(0,1) \times \mathbb{C}$. So we can conclude that all $\lambda \in \sigma(T) \setminus \left\{\frac{-1}{\alpha}\right\}$ is a normal eigenvalue of T.

As we have considered all the possibilities, the only point in the essential spectrum of $A_\alpha + B$ is $\left\{\frac{-1}{\alpha}\right\}$. $\square$

Corollary 7

$$\sigma_{ess}(A_\alpha(0)) = \left\{\frac{-1}{\alpha}\right\}.$$

Proof Notice that $A_\alpha(0) + B$ is a relatively compact perturbation of $A_\alpha(0)$. Using theorem 4 and the previous result of lemma 6, we choose $\mathcal{U} = \mathbb{C} \setminus \left\{\frac{-1}{\alpha}\right\}$ as an open set containing only normal points of $A_\alpha(0) + B$. As $A_\alpha(0)$ is a relatively compact perturbation of $A_\alpha(0) + B$, $\mathcal{U}$ contains either only eigenvalues or only normal points of $A_\alpha(0)$. As in [5] we have the characterization of the eigenvalues of $A_\alpha(0)$, we are able to say that only the second option is possible and thereby $\sigma_{ess}(A_\alpha(0)) \subseteq \left\{\frac{-1}{\alpha}\right\}$.

From that, we have two possibilities: either $\sigma_{ess}(A_\alpha(0)) = \emptyset$ or $\sigma_{ess}\,(A_\alpha(0)) = \{-1/\alpha\}$. Let us consider the first one. In this case, we could choose $\mathcal{U} = \mathbb{C}$ as the open connected set consisting only of normal points of $A_\alpha(0)$, and thinking $A_\alpha(0) + B$ as the relatively compact perturbation of $A_\alpha(0)$, the same argument as before would give us that $\sigma_{ess}\,(A_\alpha(0) + B) = \emptyset$. This is obviously false, so we can conclude that $\sigma_{ess}(A_\alpha(0)) = \left\{\frac{-1}{\alpha}\right\}$. $\square$

Now, the proof of theorem 5, which was the main aim of this section, is just a consequence of the previous corollary and an application of the same argument used in its proof.

Proof (of theorem 5). The argument of this proof is the same as in corollary 7. It can be shown that $A_\alpha(0)$ is a relatively compact perturbation of A_α for $\varepsilon > 0$. Using the results from theorem 4 and corollary 7, we can repeat the same argument used in the proof of this corollary 7. $\square$

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