ON THE ANALYTICITY OF THE
MGT-VISCOELASTIC PLATE WITH HEAT CONDUCTION

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ABSTRACT. We consider a viscoelastic plate equation of Moore-Gibson-Thompson type coupled with two different kinds of thermal laws, namely, the usual Fourier one and the heat conduction law of type III. In both cases, the resulting system is shown to generate a contraction semigroup of solutions on a suitable Hilbert space. Then we prove that these semigroups are analytic, despite the fact that the semigroup generated by the mechanical equation alone does not share the same property. This means that the coupling with the heat equation produces a regularizing effect on the dynamics, implying in particular the impossibility of the localization of solutions. As a byproduct of our main result, the exponential stability of the semigroups is established.

1. INTRODUCTION

The Moore-Gibson-Thompson (MGT) equation

$$u_{ttt} + \alpha u_{tt} + \beta A u_t + \gamma A u = 0,$$

where A is a strictly positive operator on some Hilbert space H, and $\alpha, \beta, \gamma > 0$ are given parameters, has deserved a lot of attention in recent years. Several papers have appeared in the literature on this topic (see [2, 4, 5, 6, 11, 12, 16, 19, 20], among others). The equation was originally introduced in connection with fluids mechanics [23], as a model for the acoustic velocity potential in thermally relaxing fluids (see also [11]). Surprisingly, the same equation arises as a model for the displacement in certain viscoelastic materials (see [2, 5, 7, 19] and references therein), as well as a model for the temperature displacement in a type III heat conduction with a relaxation parameter (see [3, 22]).

In the particular case where $A = \Delta^2$, with proper boundary conditions, the MGT equation appears as a possible model for the vertical displacement in viscoelastic plates (see [17]). In this sense, it is worth recalling that this equation can be obtained by considering viscosity effects of memory type when the kernel is a negative exponential. Namely, if we consider the function

$$G(s) = k^* + (\tau^{-1}k - k^*)e^{-\tau^{-1}s},$$

with $k, k^*, \tau > 0$, after combining with the local balance for the momentum equation (see e.g. [3]), we obtain

(1.1) $$\tau \varrho u_{ttt} + \varrho u_{tt} = -k^* \Delta^2 u - k \Delta^2 u_t,$$

where ϱ, τ, k, $k^* > 0$, represent the mass density, the thermal relaxation, the thermal conductivity and the conductivity of the material, respectively. This is the same as using the following constitutive equation

$$\varrho u_{tt} = -G(0)\Delta^2 u - \int_0^\infty G'(t-s)\Delta^2 u(s)\,ds,$$

with the function G defined above. We remark that (1.1) is stable if and only the dissipation condition

(1.2) $$k^*\tau < k$$

holds. In this case, the equation generates a (linear) contraction semigroup $S(t)$ of solutions (see e.g. [11]), which turns out to be exponentially stable. However, such a semigroup is not analytic [11, 16, 20].

The main goal of this paper is to show that coupling a thermal effect to equation (1.1), describing a viscoelastic plate of MGT type, makes the corresponding semigroup to be analytic, hence producing a regularizing effect on the solutions (see [13, 14] for similar results involving standard thermoelastic plates). We will see that, first, by coupling (1.1) with the Fourier heat conduction law (see e.g. [1, 17]). In this case, the coupled problem reads

$$\begin{cases} \tau\varrho u_{ttt} + \varrho u_{tt} = -k^*\Delta^2 u - k\Delta^2 u_t - m\Delta\theta, \\ c\theta_t = l\Delta\theta + m\tau\Delta u_{tt} + m\Delta u_t, \end{cases}$$

where $m \neq 0$ is the coupling parameter, $c > 0$ is the thermal capacity, and $l > 0$ is the thermal conductivity. This, in a way, could be expected. Indeed, the Fourier equation, being fully parabolic, possesses high regularization properties. At the same time, such a regularization must be transferred to the mechanical part via the coupling, which is not so obvious. Less expected, instead, is the same regularization property when in place of the Fourier law one considers heat conduction of type III, introduced by Green and Naghdi [8, 9, 10]. In this case (see e.g. [3, 13]), the coupled problem reads

$$\begin{cases} \tau\varrho u_{ttt} + \varrho u_{tt} = -k^*\Delta^2 u - k\Delta^2 u_t - m\Delta\theta, \\ c\theta_t = l^*\Delta\alpha + l\Delta\theta + m\tau\Delta u_{tt} + m\Delta u_t, \end{cases}$$

where

$$\alpha(t) = \alpha(0) + \int_0^t \theta(s)\,ds$$

represents the thermal displacement and $l^* > 0$ is the conductivity rate. Here, the thermal equation is of mixed type, and it exhibits hyperbolicity effects that prevent the instantaneous regularization of the variable α. Nonetheless, even for this kind of coupling, the resulting solution semigroup is analytic. The analysis of both models will be carried out within the dissipation condition (1.2), which is assumed to hold throughout the paper. This allows us to introduce the further parameter

$$K = k - \tau k^* > 0,$$

that will appear in the definition of the norms.

Plan of the paper. In the next Section 2, we recall some general results needed in the course of the investigation. Section 3 deals with the well-posedness of the Fourier heat conduction model, whose analyticity is established in Section 4. A similar analysis is carried out in Sections 5 and 6 for the same mechanical model, where the Fourier heat conduction law is replaced by the type III one.

2. Some Theoretical Results

In what follows, let $\mathcal{H}$ be a *complex* Hilbert space, and let $\mathbb{A} : \mathcal{H} \to \mathcal{H}$ be a densely defined linear operator of domain $\mathcal{D}(\mathbb{A}) \subset \mathcal{H}$. With standard notation, $\sigma(\mathbb{A})$ and $\rho(\mathbb{A})$ will stand for the spectrum and the resolvent of $\mathbb{A}$, respectively. Recall that $\mathbb{A}$ is said to be dissipative if

$$\mathrm{Re}\langle \mathbb{A}\mathcal{U}, \mathcal{U}\rangle_{\mathcal{H}} \leq 0, \quad \forall \mathcal{U} \in \mathcal{D}(\mathbb{A}).$$

Remark 2.1. Actually, in this paper, we will always work with an operator $\mathbb{A}$ defined on a *real* Hilbert space $\mathcal{H}$. This is not at all a problem, since all the theorems stated in the sequel apply by merely taking the standard complexification of $\mathcal{H}$ (defined as $\mathcal{H} \oplus i\mathcal{H}$), along with the standard complexification of $\mathbb{A}$. Accordingly, whenever we apply the forthcoming theorems, it is understood that we are using the complexified objects.

The first result concerns with the generation of a contraction semigroup, and is a corollary of the Lumer-Phillips theorem (see e.g. [15, 18]).

Theorem 2.2. *Assume that $\mathbb{A}$ is a closed operator. If $\mathbb{A}$ is dissipative and $\mathrm{Range}(\mathbb{A}) = \mathcal{H}$, then $\mathbb{A}$ is the infinitesimal generator of a (linear) contraction semigroup $S(t) = e^{t\mathbb{A}}$ on $\mathcal{H}$.*

Note that, as $\mathbb{A}$ is closed, the condition $\mathrm{Range}(\mathbb{A}) = \mathcal{H}$ can be equivalently stated as $0 \in \rho(\mathbb{A})$. Indeed, contrary to the Lumer-Phillips theorem, which does not require the closeness of $\mathbb{A}$ in advance, here the fact that $\mathbb{A}$ is closed is essential. However, for the kind of $\mathbb{A}$ (of differential type) used in this work, proving the closeness is completely standard.

The next theorem provides a necessary and sufficient condition for the analyticity of the semigroup (see e.g. [15]).

Theorem 2.3. *Let $\mathbb{A}$ be the infinitesimal generator of a contraction semigroup $S(t) = e^{t\mathbb{A}}$ on $\mathcal{H}$ satisfying the property*

$$(2.1) \qquad\qquad\qquad i\mathbb{R} \subset \rho(\mathbb{A}).$$

Then, the semigroup $S(t)$ is analytic if and only if

$$(2.2) \qquad\qquad \limsup_{|\lambda| \to \infty} \left\| \lambda(i\lambda I - \mathbb{A})^{-1} \right\|_{L(\mathcal{H})} < \infty.$$

Here, $L(\mathcal{H})$ denotes the Banach space of bounded linear operators on $\mathcal{H}$. A byproduct of analyticity is the impossibility of the localization of solutions.

Corollary 2.4. *Let $S(t)$ be analytic, and let $\mathcal{U} \in \mathcal{H}$. Then, if $S(t)\mathcal{U} = 0$ for some $t > 0$, it follows that $\mathcal{U} = 0$, hence $S(t)\mathcal{U} = 0$ for all times.*

Indeed, within (2.1)-(2.2), the semigroup turns out to be exponentially stable as well, namely, there exist constants $\varepsilon > 0$ and $M \geq 1$ such that

$$\|S(t)\|_{L(\mathcal{H})} \leq Me^{-\varepsilon t}.$$

This is a consequence of the famous result due to Prüss [21] reported here below.

Theorem 2.5. *A contraction semigroup $S(t) = e^{t\mathbb{A}}$ on $\mathcal{H}$ is exponentially stable if and only if* (2.1) *holds and*

$$\limsup_{|\lambda| \to \infty} \left\| (i\lambda I - \mathbb{A})^{-1} \right\|_{L(\mathcal{H})} < \infty.$$

Conditions (2.1)-(2.2) deserve some comments, that we will write in the next two remarks.

Remark 2.6. Given a contraction semigroup $S(t)$ and $\lambda \in \mathbb{R}$, if $i\lambda \in \sigma(\mathbb{A})$, then it necessarily belongs to the boundary $\partial\sigma(\mathbb{A})$ of $\sigma(\mathbb{A})$. And it is well-known that the only elements in $\sigma(\mathbb{A}) \cap \partial\sigma(\mathbb{A})$ are approximate eigenvalues. Accordingly, in order to show that $i\mathbb{R} \subset \rho(\mathbb{A})$, it is enough showing that, given any $\lambda \in \mathbb{R}$, there is no sequence $\mathcal{U}_n \in \mathcal{D}(\mathbb{A})$ of unit norm for which, in the limit $n \to \infty$,

$$i\lambda\mathcal{U}_n - \mathbb{A}\mathcal{U}_n \to 0 \text{ in } \mathcal{H}.$$

Remark 2.7. The usual strategy in order to prove analyticity is by contradiction, assuming that (2.2) fails to hold. In this case, there exist sequences λ_n, with $|\lambda_n| \to \infty$, and $\mathcal{V}_n \in \mathcal{H}$ of unit norm, such that

$$\left\| \lambda_n (i\lambda_n I - \mathbb{A})^{-1} \mathcal{V}_n \right\|_{\mathcal{H}} \to \infty, \quad \text{as } n \to \infty.$$

If we more conveniently define the vectors (of unit norm)

$$\mathcal{U}_n = \frac{\lambda_n (i\lambda_n I - \mathbb{A})^{-1} \mathcal{V}_n}{\left\| \lambda_n (i\lambda_n I - \mathbb{A})^{-1} \mathcal{V}_n \right\|_{\mathcal{H}}} \in \mathcal{D}(\mathbb{A}),$$

then the contradiction argument becomes

$$\left\| i\mathcal{U}_n - \lambda_n^{-1} \mathbb{A}\mathcal{U}_n \right\|_{\mathcal{H}} \to 0, \quad \text{as } n \to \infty,$$

which is much easier to handle.

3. The Fourier Heat Conduction Model

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a boundary sufficiently smooth (for instance, in order to guarantee the use of the divergence theorem). We analyze the MGT-viscoelastic plate coupled with Fourier heat conduction. Namely, for $\tau, \varrho, k, k^*, c, l > 0$ and $m \neq 0$, we consider the system in the unknown variables $u, \theta : \Omega \times [0, \infty) \to \mathbb{R}$

$$\begin{cases} \tau\varrho u_{ttt} + \varrho u_{tt} = -k^*\Delta^2 u - k\Delta^2 u_t - m\Delta\theta, \\ c\theta_t = l\Delta\theta + m\tau\Delta u_{tt} + m\Delta u_t, \end{cases}$$

subject to the initial conditions

$$u(0) = u_0, \quad u_t(0) = v_0, \quad u_{tt}(0) = w_0, \quad \theta(0) = \theta_0,$$

where u_0, v_0, w_0, θ_0 are prescribed data. The system is supplemented with the boundary conditions

$$u(x, t) = \frac{\partial u}{\partial \nu}(x, t) = \theta(x, t) = 0, \quad x \in \partial\Omega.$$

In particular, we are here assuming the so-called *clamped* boundary conditions for the plate.

In order to frame the system in the correct functional setting, the first step is defining a suitable Hilbert space accounting for the boundary conditions above. To this end, we consider the usual Hilbert space $(L^2(\Omega), \|\cdot\|, \langle\cdot, \cdot\rangle)$, along with the standard Sobolev spaces $H^p(\Omega)$, $H_0^1(\Omega)$ and $H_0^2(\Omega)$. Recall that[1]

$$\|u\|_{H_0^1} = \|\nabla u\| \qquad \text{and} \qquad \|u\|_{H_0^2} = \|\Delta u\|.$$

Finally, we set (in what follows, Ω will be omitted for short)

$$\mathcal{H} = H_0^2 \times H_0^2 \times L^2 \times L^2.$$

Hence, introducing the state vector

$$\mathcal{U}(t) = (u(t), v(t), w(t), \theta(t)),$$

we are able to rewrite our system as the ODE in $\mathcal{H}$

$$\frac{d}{dt}\mathcal{U}(t) = \mathbb{A}\mathcal{U}(t),$$

with initial condition

$$\mathcal{U}(0) = (u_0, v_0, w_0, \theta_0) \in \mathcal{H}.$$

Here, $\mathbb{A}$ is the linear operator given by

$$\mathbb{A}\begin{pmatrix} u \\ v \\ w \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ w \\ -\frac{1}{\tau\varrho}\left[\varrho w + \Delta(k^*\Delta u + k\Delta v) + m\Delta\theta\right] \\ \frac{1}{c}\Delta\left[l\theta + m\tau w + mv\right] \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathbb{A}) = \left\{ \mathcal{U} \in \mathcal{H} \;\middle|\; \begin{array}{l} w \in H_0^2 \\ \theta \in H^2 \cap H_0^1 \\ k^*\Delta u + k\Delta v \in H^2 \end{array} \right\}.$$

It is easily seen by direct computations that the operator $\mathbb{A}$, besides being densely defined, is closed as well. This is essential in the proof of the next result, based on an application of Theorem 2.2.

Theorem 3.1. *The operator $\mathbb{A}$ is the infinitesimal generator of a strongly continuous linear semigroup $S(t) = e^{t\mathbb{A}}$ on the phase space $\mathcal{H}$. Besides, $S(t)$ is a contraction with respect to the (equivalent) norm of $\mathcal{H}$*

$$\|(u, v, w, \theta)\|_{\mathcal{H}}^2 = \varrho\|v + \tau w\|^2 + k^*\|\Delta u + \tau\Delta v\|^2 + \tau K\|\Delta v\|^2 + c\|\theta\|^2,$$

where we recall that $K = k - \tau k^* > 0$.

In particular, for every initial datum $\mathcal{U}_0 = (u_0, v_0, w_0, \theta_0) \in \mathcal{H}$, there exists a unique solution $\mathcal{U} \in C([0, \infty), \mathcal{H})$ to the Cauchy problem above, given by

$$\mathcal{U}(t) = S(t)\mathcal{U}_0.$$

The result will be proved by applying the abstract Theorem 2.2. Indeed, the operator $\mathbb{A}$ is dissipative as, for $\mathcal{U} = (u, v, w, \alpha, \theta)$, direct calculations give

(3.1) $$\operatorname{Re}(\mathbb{A}\mathcal{U}, \mathcal{U})_{\mathcal{H}} = -K\|\Delta v\|^2 - l\|\nabla\theta\|^2 \le 0.$$

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[1] As usual, abusing the notation, we keep writing $\|\nabla u\|$ to mean the norm of ∇u in $[L^2(\Omega)]^2$.

Accordingly, the proof of Theorem 3.1 follows from the next lemma.

Lemma 3.2. *We have* $\mathrm{Range}(\mathbb{A}) = \mathcal{H}$.

Proof. For $\mathcal{F} = (f, g, h, q) \in \mathcal{H}$ arbitrarily given, we want to find $\mathcal{U} = (u, v, w, \theta) \in \mathcal{D}(\mathbb{A})$ satifying

$$\mathbb{A}\mathcal{U} = \mathcal{F}.$$

Equivalently,

$$\begin{cases} v = f, \\ w = g, \\ -\Delta(k^*\Delta u - k\Delta v) - \varrho w - m\Delta\theta = \tau\varrho h, \\ m\Delta v + m\tau\Delta w + l\Delta\theta = cq. \end{cases}$$

Plugging f and g in the last two equations, we end up with

$$\begin{cases} \Delta^2 z = -\varrho\tau h - m\Delta\theta, \\ -l\Delta\theta = m\Delta f + m\tau\Delta g - cq, \end{cases}$$

having set

$$z = k^*u + kf.$$

We first solve the second equation. Since the right-hand side belongs to L^2, this is immediately done by inverting $-\Delta$, viewed as an operator on L^2 with domain $H^2 \cap H_0^1$, so obtaining

$$\theta = \frac{1}{l}(-\Delta)^{-1}\big[m\Delta f + m\tau\Delta g - cq\big] \in H^2 \cap H_0^1.$$

At this point, $\Delta\theta$ is now a known vector of L^2. Hence, calling $y = -\varrho\tau h - m\Delta\theta \in L^2$, the first equation becomes

$$\Delta^2 z = y.$$

Then, we can invert Δ^2, viewed as an operator on L^2 with domain $H^4 \cap H_0^2$. Accordingly,

$$z = (\Delta^2)^{-1}y \in H^4 \cap H_0^2.$$

In particular, as $v = f \in H_0^2$, we learn that $u \in H_0^2$ and $k^*\Delta u + k\Delta v \in H^2$. $\qquad\square$

4. Analyticity of the Fourier Heat Conduction Model

We now show that the effect of the coupling between the MGT-viscoelastic equation and heat conduction of Fourier type renders analytic the corresponding solution semigroup. Besides, the coupled dynamics remains exponentially stable.

Theorem 4.1. *The semigroup* $S(t) = e^{t\mathbb{A}}$ *on* $\mathcal{H}$ *is analytic and exponentially stable.*

Remark 4.2. In particular, due to Corollary 2.4, we have that if $S(t)\mathcal{U}_0 = 0$ for some $t > 0$, then $S(t)\mathcal{U}_0 = 0$ for all times.

The proof of Theorem 4.1 will be carried out by exploiting Theorem 2.3 (implying in turn Theorem 2.5, that ensures the exponential stability of the semigroup). This amounts to proving the next two lemmas.

Lemma 4.3. *The operator* $\mathbb{A}$ *satisfies condition* (2.1)*, that is,* $i\mathbb{R} \subset \rho(\mathbb{A})$.

Proof. On account of Remark 2.6, and since we already know that $0 \in \varrho(\mathbb{A})$, it is enough showing that, given any $\lambda \neq 0$, there is no sequence $\mathcal{U}_n = (u_n, v_n, w_n, \theta_n) \in \mathcal{D}(\mathbb{A})$ of unit $\mathcal{H}$-norm for which

$$(4.1) \qquad i\lambda \mathcal{U}_n - \mathbb{A}\mathcal{U}_n \to 0 \ \text{ in } \mathcal{H}.$$

By contradiction, let us then assume the existence of such a $\mathcal{U}_n$. Multiplying (4.1) by $\mathcal{U}_n$ in $\mathcal{H}$, and using the dissipation inequality (3.1), we infer that

$$\Delta v_n, \nabla \theta_n \to 0 \ \text{ in } L^2.$$

At this point, writing (4.1) componentwise, we obtain four relations, two of which read

$$i\lambda u_n - v_n \to 0 \ \text{ in } H_0^2,$$
$$i\lambda v_n - w_n \to 0 \ \text{ in } H_0^2.$$

Hence, we immediately conclude that

$$u_n \to 0 \ \text{ in } H_0^2, \qquad w_n \to 0 \ \text{ in } H_0^2,$$

meaning that $\mathcal{U}_n \to 0$ in $\mathcal{H}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 4.4. *The operator $\mathbb{A}$ satisfies condition* (2.2).

Proof. By contradiction, and with an eye to Remark 2.7, let us assume the existence of a sequence $\lambda_n \in \mathbb{R}$, with $|\lambda_n| \to \infty$, and a sequence of vectors $\mathcal{U}_n = (u_n, v_n, w_n, \theta_n) \in \mathcal{D}(\mathbb{A})$ of unit $\mathcal{H}$-norm such that

$$(4.2) \qquad i\mathcal{U}_n - \lambda_n^{-1}\mathbb{A}\mathcal{U}_n \to 0 \ \text{ in } \mathcal{H}.$$

Written in components,

$$(4.3) \qquad iu_n - \lambda_n^{-1}v_n \to 0 \ \text{ in } H_0^2,$$

$$(4.4) \qquad iv_n - \lambda_n^{-1}w_n \to 0 \ \text{ in } H_0^2,$$

$$(4.5) \qquad i\tau\varrho w_n + \lambda_n^{-1}\Delta(k^*\Delta u_n + k\Delta v_n) + \varrho\lambda_n^{-1}w_n + m\lambda_n^{-1}\Delta\theta_n \to 0 \ \text{ in } L^2,$$

$$(4.6) \qquad ic\theta_n - m\lambda_n^{-1}\Delta v_n - m\tau\lambda_n^{-1}\Delta w_n - l\lambda_n^{-1}\Delta\theta_n \to 0 \ \text{ in } L^2.$$

Since v_n is bounded in H_0^2 and $|\lambda_n| \to \infty$, we immediately infer from (4.3) that

$$(4.7) \qquad u_n \to 0 \ \text{ in } H_0^2.$$

Multiplying (4.2) by $\mathcal{U}_n$ in $\mathcal{H}$, and exploiting the dissipation inequality (3.1), we deduce that

$$\lambda_n^{-1/2}\nabla\theta_n \to 0 \ \text{ in } L^2.$$

Thanks to the Gagliardo-Nirenberg interpolation inequality,

$$\left\| \lambda_n^{-1/2}\nabla w_n \right\|^2 \leq C\lambda_n^{-1}\left\| \Delta w_n \right\| \left\| w_n \right\|,$$

for some suitable $C > 0$. Since w_n and $\lambda_n^{-1}\Delta w_n$ are both bounded in L^2, due to (4.4), we deduce the control

$$\sup_n \lambda_n^{-1/2}\|\nabla w_n\| < \infty.$$

Using the relations above, we now take the inner product of (4.6) and θ_n, to get

$$ic\left\| \theta_n \right\|^2 + m\tau\langle \lambda_n^{-1/2}\nabla w_n, \lambda_n^{-1/2}\nabla\theta_n\rangle \to 0,$$

and since $\lambda_n^{-1/2}\nabla w_n$ is bounded in L^2, we immediately conclude that

$$\theta_n \to 0 \ \text{in} \ L^2. \tag{4.8}$$

At this point, we denote

$$\tilde{v}_n = v_n + k^* k^{-1} u_n.$$

Using (4.5) and (4.6) and the convergences above, we can write

$$i\tau w_n + k\lambda_n^{-1}\Delta^2\tilde{v}_n + m\lambda_n^{-1}\Delta\theta_n \to 0 \ \text{in} \ L^2, \tag{4.9}$$

and

$$im\tau\Delta\tilde{v}_n + l\lambda_n^{-1}\Delta\theta_n \to 0 \ \text{in} \ L^2. \tag{4.10}$$

We claim that

$$\sup_n \lambda_n^{-1/2}\|\nabla\Delta\tilde{v}_n\| < \infty. \tag{4.11}$$

Indeed, $\Delta\tilde{v}_n$ is bounded in L^2 by construction, thus from (4.10) $\lambda_n^{-1}\Delta\theta_n$ is bounded in L^2 as well. Accordingly, by (4.9),

$$\sup_n \lambda_n^{-1}\|\Delta^2\tilde{v}_n\| < \infty.$$

Applying once again the Gagliardo-Nirenberg inequality,

$$\|\lambda_n^{-1/2}\nabla\Delta\tilde{v}_n\|^2 \leq C\lambda_n^{-1}\|\Delta^2\tilde{v}_n\|\|\Delta\tilde{v}_n\|,$$

establishing (4.11). We can now test (4.10) by $\Delta\tilde{v}_n$, to obtain

$$im\tau\|\Delta\tilde{v}_n\|^2 - l\langle\lambda_n^{-1/2}\nabla\theta_n, \lambda_n^{-1/2}\nabla\Delta\tilde{v}_n\rangle + l\lambda_n^{-1}\int_{\partial\Omega}\frac{\partial\theta_n}{\partial\nu}\Delta\tilde{v}_n \to 0. \tag{4.12}$$

The next step is showing that the latter term vanishes. To this end, we first write

$$\lambda_n^{-1}\left|\int_{\partial\Omega}\frac{\partial\theta_n}{\partial\nu}\Delta\tilde{v}_n\right| \leq \lambda_n^{-3/4}\left\|\frac{\partial\theta_n}{\partial\nu}\right\|_{L^2(\partial\Omega)}\lambda_n^{-1/4}\|\Delta\tilde{v}_n\|_{L^2(\partial\Omega)}.$$

Using the trace theorem and the Sobolev spaces inequalities (see e.g. [15, Theorem 1.4.4]), along with the previous relations, we have (in what follows $C > 0$ stands for a generic constant)[2]

$$\lambda_n^{-3/4}\left\|\frac{\partial\theta_n}{\partial\nu}\right\|_{L^2(\partial\Omega)} \leq C\left(\lambda_n^{-1/2}\|\nabla\theta_n\|\right)^{1/2}\left(\lambda_n^{-1}\|\Delta\theta_n\|\right)^{1/2} \to 0,$$

and, with the aid of the Young inequality,

$$\lambda_n^{-1/4}\|\Delta\tilde{v}_n\|_{L^2(\partial\Omega)} \leq C\left(\lambda_n^{-1/2}\|\Delta\tilde{v}_n\|_{H^1}\right)^{1/2}\|\Delta\tilde{v}_n\|^{1/2}$$
$$\leq C\lambda_n^{-1/2}\|\nabla\Delta\tilde{v}_n\| + C\|\Delta\tilde{v}_n\|,$$

which is bounded by (4.11). We conclude that, as desired,

$$l\lambda_n^{-1}\int_{\partial\Omega}\frac{\partial\theta_n}{\partial\nu}\Delta\tilde{v}_n \to 0.$$

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[2]Recall that, as $\theta \in H^2 \cap H_0^1$, the H^2-norm of θ is equivalent to $\|\Delta\theta\|$.

Therefore, (4.12) becomes

$$im\tau \, \|\Delta\tilde{v}_n\|^2 - l\langle \lambda_n^{-1/2}\nabla\theta_n, \lambda_n^{-1/2}\nabla\Delta\tilde{v}_n\rangle \to 0.$$

Recalling again that $\lambda_n^{-1/2}\|\nabla\theta_n\| \to 0$ and invoking (4.11), it is apparent that $\Delta\tilde{v}_n$ tends to zero in L^2, which in turn implies

(4.13) $$v_n \to 0 \ \text{in} \ H_0^2.$$

We are left to prove that

(4.14) $$w_n \to 0 \ \text{in} \ L^2.$$

To see that, we multiply (4.9) by w_n in L^2. We get

$$i\tau\|w_n\|^2 + k\langle \Delta\tilde{v}_n, \lambda_n^{-1}\Delta w_n\rangle + \varrho\lambda_n^{-1}\|w_n\|^2 + m\langle \lambda_n^{-1}\Delta\theta_n, w_n\rangle \to 0.$$

Knowing that $\Delta\tilde{v}_n \to 0$ in L^2, we learn from (4.10) that $\lambda_n^{-1}\Delta\theta_n \to 0$ in L^2. Exploiting once more the fact that $\lambda_n^{-1}\Delta w_n$ is bounded in light of (4.4), we finally arrive at $w_n \to 0$ in L^2.

Collecting (4.7), (4.8), (4.13) and (4.14), we have that $\|\mathcal{U}_n\|_{\mathcal{H}} \to 0$ in $\mathcal{H}$, which contradicts the fact that $\|\mathcal{U}_n\|_{\mathcal{H}} = 1$. The result is proved. $\qquad\square$

Remark 4.5. As a final comment, we mention that other physically relevant boundary conditions for the mechanical part could be also considered. For instance, we could assume the *hinged* boundary conditions for the plate, namely, $u = \Delta u = 0$ on the boundary $\partial\Omega$. In fact, in this situation the analysis becomes easier, and all the argument above can be immediately adapted. In particular, the treatment of the boundary conditions is similar to the one proposed in the next two sections for the case of type III heat conduction.

5. The Type III Heat Conduction Model

An alternative theory to the Fourier heat conduction is the type III one (see [3, 8, 9, 10]). In this case, given $\tau, \varrho, k, k^*, c, l, l^* > 0$ and $m \neq 0$, we have the system the unknown variables $u, \alpha : \Omega \times [0, \infty) \to \mathbb{R}$

$$\begin{cases} \tau\varrho u_{ttt} + \varrho u_{tt} = -k^*\Delta^2 u - k\Delta^2 u_t - m\Delta\alpha_t, \\ c\alpha_{tt} = l^*\Delta\alpha + l\Delta\alpha_t + m\tau\Delta u_{tt} + m\Delta u_t, \end{cases}$$

subject to the initial conditions

$$u(0) = u_0, \quad u_t(0) = v_0, \quad u_{tt}(0) = w_0, \quad \alpha(0) = \alpha_0, \quad \alpha_t(0) = \theta_0,$$

where $u_0, v_0, w_0, \alpha_0, \theta_0$ are prescribed data. The system is supplemented with the boundary conditions

$$u(x, t) = \Delta u(x, t) = \alpha(x, t) = 0, \quad x \in \partial\Omega.$$

In particular, we are here assuming the *hinged* boundary conditions for the plate.

Setting $\mathsf{H} = L^2(\Omega)$, we introduce the strictly positive selfadjoint operator $A : \mathsf{H} \to \mathsf{H}$ as

$$A = -\Delta \quad \text{with domain} \quad \mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega).$$

Accordingly, denoting by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the standard scalar product and norm in H, respectively, for any $r > 0$ we set

$$\mathsf{H}^r = \mathcal{D}(A^{r/2}),$$

endowed with the scalar product and norm

$$\langle u, v \rangle_r = \langle A^{r/2}u, A^{r/2}v \rangle \qquad \text{and} \qquad \|u\|_r = \|A^{r/2}u\|.$$

In particular, $\mathsf{H}^1 = H_0^1(\Omega)$ and $\mathsf{H}^2 = H_0^1(\Omega) \cap H^2(\Omega)$. Next, we consider the product Hilbert space

$$\mathcal{H} = \mathsf{H}^2 \times \mathsf{H}^2 \times \mathsf{H} \times \mathsf{H}^1 \times \mathsf{H}.$$

Accordingly, we rewrite our system in the abstract form

$$\begin{cases} \tau\varrho u_{ttt} + \varrho u_{tt} = -A(k^*Au + kAu_t - m\alpha_t), \\ c\alpha_{tt} = -A(l^*\alpha + l\alpha_t + m\tau u_{tt} + mu_t). \end{cases}$$

Introducing the state vector

$$\mathcal{U}(t) = (u(t), v(t), w(t), \alpha(t), \theta(t)),$$

we obtain the ODE in $\mathcal{H}$

$$\frac{d}{dt}\mathcal{U}(t) = \mathbb{A}\mathcal{U}(t),$$

with initial condition

$$\mathcal{U}(0) = (u_0, v_0, w_0, \alpha_0, \theta_0) \in \mathcal{H}.$$

Here, $\mathbb{A}$ is the linear operator given by

$$\mathbb{A}\begin{pmatrix} u \\ v \\ w \\ \alpha \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ w \\ -\frac{1}{\tau\varrho}\big[\varrho w + A(k^*Au + kAv - m\theta)\big] \\ \theta \\ -\frac{1}{c}A\big[l^*\alpha + l\theta + m\tau w + mv\big] \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathbb{A}) = \left\{ \mathcal{U} \in \mathcal{H} \;\middle|\; \begin{array}{l} w \in \mathsf{H}^2 \\ \theta \in \mathsf{H}^1 \\ k^*Au + kAv - m\theta \in \mathsf{H}^2 \\ l^*\alpha + l\theta \in \mathsf{H}^2 \end{array} \right\}.$$

As in the previous case, the operator $\mathbb{A}$ is densely defined and is closed.

Remark 5.1. Although we wrote $\Delta u = 0$ on $\partial\Omega$, as commonly done, the domain of the operator actually dictates the correct boundary conditions for u, that should be more properly given in the form $k^*Au + kAu_t = 0$ on $\partial\Omega$. By the same token, it is implicitly assumed that $\alpha_t = 0$ on $\partial\Omega$.

Theorem 5.2. *The operator $\mathbb{A}$ is the infinitesimal generator of a strongly continuous linear semigroup $S(t) = e^{t\mathbb{A}}$ on the phase space $\mathcal{H}$. Besides, $S(t)$ is a contraction with respect to the (equivalent) norm of $\mathcal{H}$*

$$\|(u, v, w, \alpha, \theta)\|_{\mathcal{H}}^2 = \varrho\|v + \tau w\|^2 + k^*\|u + \tau v\|_2^2 + \tau K\|v\|_2^2 + c\|\theta\|^2 + l^*\|\alpha\|_1^2,$$

where we recall that $K = k - \tau k^* > 0$.

Similarly to the Fourier case, the operator $\mathbb{A}$ is dissipative. Indeed, for $\mathcal{U} = (u, v, w, \alpha, \theta)$, by direct calculations we have

$$(5.1) \qquad \operatorname{Re}(\mathbb{A}\mathcal{U}, \mathcal{U})_{\mathcal{H}} = -K\|v\|_2^2 - l\|\theta\|_1^2 \leq 0.$$

Accordingly, as $\mathbb{A}$ is closed, and with reference to Theorem 2.2, the proof of Theorem 5.2 follows from the next lemma.

Lemma 5.3. *We have* $\operatorname{Range}(\mathbb{A}) = \mathcal{H}$.

Proof. Given any $\mathcal{F} = (f, g, h, r, q) \in \mathcal{H}$, we look for a solution $(u, v, w, \alpha, \theta) \in \mathcal{D}(\mathbb{A})$ to the system

$$\begin{cases} v = f, \\ w = g, \\ A(k^*Au + kAv - m\theta) + \varrho w = -\tau\varrho h, \\ \theta = r, \\ A(l^*\alpha + l\theta) + m\tau Aw + mAv = -cq. \end{cases}$$

Plugging $w = g \in \mathsf{H}^2$ into the third equation, we obtain

$$A(k^*Au + kAv - m\theta) = -\varrho g - \tau\varrho h.$$

Note that the right-hand side (call it b) belongs to H, so the equation

$$Ax = b$$

has a unique solution $x = A^{-1}b \in \mathsf{H}^2$. Accordingly, we look for u such that

$$k^*Au + kAv - m\theta = x.$$

Taking advantage of the equalities $v = f \in \mathsf{H}^2$ and $\theta = r \in \mathsf{H}^1$, we readily obtain

$$u = \frac{1}{k^*}A^{-1}\big(-kAf + mr + x\big) \in \mathsf{H}^2.$$

Once u is found, $k^*Au + kAv - m\theta \in \mathsf{H}^2$ by construction. Analogously, the last equation reads

$$A(l^*\alpha + l\theta) = \hat{b},$$

where

$$\hat{b} = -m\tau Ag - mAf - cq \in \mathsf{H}.$$

Hence, $\hat{x} = A^{-1}\hat{b} \in \mathsf{H}^2$, and using once more the equality $\theta = r$ we end up with

$$\alpha = \frac{1}{l^*}(\hat{x} - lr) \in \mathsf{H}^1.$$

Again, the relation $l^*\alpha + l\theta \in \mathsf{H}^2$ follows by construction. $\qquad\square$

6. Analyticity of the Type III Heat Conduction Model

As in the Fourier case, the coupling of the MGT-viscoelastic plate equation with the type III heat conduction produces a regularizing effect on the solutions. Indeed, the following theorem holds.

Theorem 6.1. *The semigroup $S(t) = e^{t\mathbb{A}}$ on $\mathcal{H}$ is analytic and exponentially stable.*

Remark 6.2. As in the previous case (cf. Corollary 2.4), this implies the impossibility of the localization of solutions.

On account of Theorem 2.3 (and the subsequent Theorem 2.5), the proof of Theorem 6.1 is a consequence of the next two lemmas.

Lemma 6.3. *The operator $\mathbb{A}$ satisfies condition (2.1), that is, $i\mathbb{R} \subset \rho(\mathbb{A})$.*

Proof. As in the proof of Lemma 4.3, assume by contradiction the existence of $\lambda \neq 0$ along with a sequence $\mathcal{U}_n = (u_n, v_n, w_n, \alpha_n, \theta_n) \in \mathcal{D}(\mathbb{A})$ of unit $\mathcal{H}$-norm such that

$$(6.1) \qquad i\lambda \mathcal{U}_n - \mathbb{A}\mathcal{U}_n \to 0 \ \text{in} \ \mathcal{H}.$$

Multiplying (6.1) by $\mathcal{U}_n$ in $\mathcal{H}$, and using the dissipation inequality (5.1), we infer that

$$v_n \to 0 \ \text{in} \ \mathsf{H}^2, \qquad \theta_n \to 0 \ \text{in} \ \mathsf{H}^1.$$

Besides, by (6.1) componentwise, we get

$$i\lambda u_n - v_n \to 0 \ \text{in} \ \mathsf{H}^2,$$
$$i\lambda v_n - w_n \to 0 \ \text{in} \ \mathsf{H}^2,$$
$$i\lambda \alpha_n - \theta_n \to 0 \ \text{in} \ \mathsf{H}^1.$$

Therefore,

$$u_n \to 0 \ \text{in} \ \mathsf{H}^2, \qquad w_n \to 0 \ \text{in} \ \mathsf{H}^2, \qquad \alpha_n \to 0 \ \text{in} \ \mathsf{H}^1,$$

meaning that $\mathcal{U}_n \to 0$ in $\mathcal{H}$. $\qquad\square$

Lemma 6.4. *The operator $\mathbb{A}$ satisfies condition (2.2).*

Proof. In the same spirit of the proof of Lemma 4.4, by contradiction let $\lambda_n \in \mathbb{R}$, with $|\lambda_n| \to \infty$, and $\mathcal{U}_n = (u_n, v_n, w_n, \alpha_n, \theta_n) \in \mathcal{D}(\mathbb{A})$, with $\|\mathcal{U}_n\|_{\mathcal{H}} = 1$, satisfy

$$(6.2) \qquad i\mathcal{U}_n - \lambda_n^{-1}\mathbb{A}\mathcal{U}_n \to 0 \ \text{in} \ \mathcal{H}.$$

Multiplying (6.2) by $\mathcal{U}_n$ in $\mathcal{H}$, we learn from (5.1) that

$$(6.3) \qquad \lambda_n^{-1/2}\theta_n \to 0 \ \text{in} \ \mathsf{H}^1.$$

Writing (6.2) componentwise, we get

$$(6.4) \qquad iu_n - \lambda_n^{-1}v_n \to 0 \ \text{in} \ \mathsf{H}^2,$$

$$(6.5) \qquad iv_n - \lambda_n^{-1}w_n \to 0 \ \text{in} \ \mathsf{H}^2,$$

$$(6.6) \qquad i\tau\varrho w_n + \lambda_n^{-1}A(k^*Au_n + kAv_n) + \varrho\lambda_n^{-1}w_n - m\lambda_n^{-1}A\theta_n \to 0 \ \text{in} \ \mathsf{H},$$

$$(6.7) \qquad i\alpha_n - \lambda_n^{-1}\theta_n \to 0 \ \text{in} \ \mathsf{H}^1,$$

$$(6.8) \qquad ic\theta_n + m\lambda_n^{-1}Av_n + m\tau\lambda_n^{-1}Aw_n + l\lambda_n^{-1}A\theta_n + l^*\lambda_n^{-1}A\alpha_n \to 0 \ \text{in} \ \mathsf{H}.$$

Since v_n is bounded in H^2, we see at once from (6.4) that
$$u_n \to 0 \ \text{in} \ \mathsf{H}^2.$$
Besides, in light of (6.3) and (6.7),
$$\alpha_n \to 0 \ \text{in} \ \mathsf{H}^1.$$
At this point, exploiting the information obtained so far, the inner product of (6.8) with θ_n gives
$$ic\|\theta_n\|^2 + m\tau\langle\lambda_n^{-1/2}w_n, \lambda_n^{-1/2}\theta_n\rangle_1 \to 0.$$
On the other hand, due to (6.5), the term $\lambda_n^{-1/2}w_n$ is bounded in H^1, for
$$\lambda_n^{-1}\|w_n\|_1^2 \leq \lambda_n^{-1}\|w_n\|_2\|w_n\|.$$
Hence, using again (6.3), we infer that
$$\theta_n \to 0 \ \text{in} \ \mathsf{H}.$$
At this point, we denote
$$\tilde{v}_n = v_n + k^*k^{-1}u_n,$$
and we claim that
(6.9)
$$\sup_n \lambda_n^{-1}\|\tilde{v}_n\|_3^2 < \infty.$$
Indeed, observing that $\lambda_n^{-1}w_n \to 0$ in H (as w_n is bounded in H), we can rewrite (6.6) in the form
(6.10)
$$i\tau\varrho w_n + k\lambda_n^{-1}A^2\tilde{v}_n - m\lambda_n^{-1}A\theta_n \to 0 \ \text{in} \ \mathsf{H},$$
and a multiplication by $A\tilde{v}_n$ (which is bounded in H) yields
$$i\tau\varrho\langle w_n, A\tilde{v}_n\rangle + k\lambda_n^{-1}\|\tilde{v}_n\|_3^2 - m\langle\lambda_n^{-1/2}\theta_n, \lambda_n^{-1/2}A\tilde{v}_n\rangle_1 \to 0.$$
The first term above is clearly bounded. Moreover, we have the estimate
$$m|\langle\lambda_n^{-1/2}\theta_n, \lambda_n^{-1/2}A\tilde{v}_n)\rangle_1| \leq \frac{k}{2}\lambda_n^{-1}\|\tilde{v}_n\|_3^2 + \frac{m^2}{2k}\lambda_n^{-1}\|\theta_n\|_1^2,$$
where the latter term goes to zero in light of (6.3). This establishes the sought estimate (6.9). We are now in a position to handle (6.8). Exploiting the convergence $\theta_n \to 0$ in H along with (6.4)-(6.5), we transform (6.8) into
$$imAu_n + im\tau Av_n + l\lambda_n^{-1}A\theta_n + l^*\lambda_n^{-1}A\alpha_n \to 0 \ \text{in} \ \mathsf{H}.$$
On the other hand, $Av_n = A\tilde{v}_n - k^*k^{-1}Au_n$, and recalling that $u_n \to 0$ in H^2, the convergence above can be equivalently written as
$$im\tau A\tilde{v}_n + l\lambda_n^{-1}A\theta_n + l^*\lambda_n^{-1}A\alpha_n \to 0 \ \text{in} \ \mathsf{H}.$$
Now, multiplying by $A\tilde{v}_n$, we obtain
$$im\tau\|\tilde{v}_n\|_2^2 + l^*\langle\lambda_n^{-1/2}\alpha_n, \lambda_n^{-1/2}A\tilde{v}_n\rangle_1 + l\langle\lambda_n^{-1/2}\theta_n, \lambda_n^{-1/2}A\tilde{v}_n\rangle_1 \to 0.$$
Since both α_n and $\lambda_n^{-1/2}\theta_n$ go to zero in H^1, taking advantage of (6.9) we conclude that $\tilde{v}_n \to 0$ in H^2, in turn implying (as we already know that $u_n \to 0$ in H^2)
$$v_n \to 0 \ \text{in} \ \mathsf{H}^2.$$

The last step is showing the convergence to zero of w_n in H. But this can be easily drawn by multiplying (6.10) by w_n, giving

$$i\varrho\|w_n\|^2 + k\langle A\tilde{v}_n, \lambda_n^{-1}Aw_n\rangle - m\langle\theta_n, \lambda_n^{-1}Aw_n\rangle \to 0 \quad \text{in } \mathsf{H}.$$

By (6.5) we know that $\lambda_n^{-1}Aw_n$ is bounded in H, and the convergences $A\tilde{v}_n, \theta_n \to 0$ in H readily imply

$$w_n \to 0 \ \text{ in } \mathsf{H}.$$

In summary, we proved that $u_n, v_n \to 0$ in H^2, $\alpha_n \to 0$ in H^1 and $w_n, \theta_n \to 0$ in H, meaning that $\mathcal{U}_n \to 0$ in $\mathcal{H}$. This contradicts the assumption $\|\mathcal{U}_n\|_{\mathcal{H}} = 1$. □

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