ON UNIQUENESS AND INSTABILITY FOR THE MOORE-GIBSON-THOMPSON THERMOELASTICITY

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Abstract. It is known that in the case that several constitutive tensors fail to be positive definite the system of the thermoelasticity could become unstable and, in certain cases, ill-posed in the sense of Hadamard. In this paper we consider the Moore-Gibson-Thompson thermoelasticity in the case that some of the constitutive tensors fail to be positive and we will prove basic results concerning uniqueness and instability of solutions. We first consider the case of the heat conduction when dissipation condition holds but some constitutive tensors can fail to be positive. In this case we prove the uniqueness and instability by means of the logarithmic convexity argument. Second we study the thermoelastic system only assuming that the thermal conductivity tensor and the mass density are positive and we obtain the uniqueness of solutions by means of the Lagrange identities method. By the logarithmic convexity argument we prove later the instability of solutions whenever the elasticity tensor fails to be positive, but assuming that the conductivity rate is positive and the thermal dissipation condition hold. We also sketch similar results when conductivity rate and/or the thermal conductivity fail to be positive definite, but the elasticity tensor is positive definite and the dissipation condition holds. Last sections are devoted to consider the case when a third order equation is proposed for the displacement (which comes from the viscoelasticiy). A similar study is sketched in these cases.

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1. Introduction

The Moore-Gibson-Thompson (MGT) equation

$$u_{ttt} + \alpha u_{tt} + \beta A u_t + \gamma A u = 0, \tag{1.1}$$

where A is a strictly positive operator on some Hilbert space and $\alpha, \beta, \gamma > 0$ are given parameters, has deserved much attention in recent years, with several papers that have appeared in the literature on this topic (see [4, 6, 7, 8, 14, 19, 22, 23, 24], among others). The model has been originally introduced in connection with fluids mechanics [28].

It is worth recalling that recently this equation has been obtained by introducing a relaxation parameter into the type III heat conduction[1]. Therefore it is also natural to consider it as a heat conduction equation. This proposition has been considered recently in [5, 26] in order to consider the Moore-Gibson-Thompson thermoelaticity. In this paper we are going to consider this theory. At the same time it can be obtained as a particular case for the three-phase-lag theory proposed by Choudhuri [3].

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[1]This is motivated becase type III heat conduction violates the principle of causality see [10, 27]

On the other side it is well known that when the elasticity tensor fails to be positive definite we cannot expect stability of the solutions for several thermoelastic theories [16, 17]. In this paper we want to study the uniqueness and instability of solutions in the case that certain thermoelastic tensors fail to be positive definite. Our main tools are the logarithmic convexity argument and the Lagrange identities method (see [15], for instance).

The axioms of thermomechanics imply that the thermal conductivity tensor cannot have negative sign. But these axioms do not give other conditions on any of the remaining tensors, in particular the elasticity tensor [18, 13]. For real elastic materials initially prestressed, the elasticity tensor does not have necessarily positive sign [11, 12, 13]. Hence, it is needed to analyze the problem determined by thermoelastic systems when the elasticity tensor is not positive definite (see, for example, [2, 15, 16, 17, 20, 29]). It is worth noting that the problem can be ill posed in the sense of Hadamard. Hence the task to deal with this problem can be difficult and it is not easy to clarify the qualitative properties of the solutions. Results on uniqueness, instability, continuous dependence in the sense of Holder, structural stability, etc., have been obtained for different situations (see [1, 16, 17, 21, 25]). Although the techniques used to prove them are quite standard we highlight that few attention has been considered in case of third order in time equations or systems. As in this paper we consider equations and/or systems of third order in time we believe that this contribution is of interest in the study of the thermoelasticity.

In this paper we always consider a three dimensional bounded region B such that its boundary is smooth enough to apply the divergence theorem. Also, in all the sections we assume that $k_{ij}(\mathbf{x})$ (thermal conductivity tensor) and the tensor $k^*_{ij}(\mathbf{x})$ given in sections below are symmetric tensors, that is

$$k_{ij}(\mathbf{x}) = k_{ji}(\mathbf{x}), \quad k^*_{ij}(\mathbf{x}) = k^*_{ji}(\mathbf{x}), \quad \mathbf{x} \in B. \tag{1.2}$$

and that

i) the thermal relaxation parameter τ (see sections below) is a positive number
ii) all the functions defined in the systems given in the sections below are bounded

This paper is organized as follows. Section 2 is devoted to prove the uniqueness and instability for the heat conduction of MGT type when dissipation condition holds but some constitutive tensors can fail to be positive. Sections 3 and 4 are devoted to the uniquess and instability for the MGT thermoelastic system only assuming that the thermal conductivity tensor and the mass density are positive (for the uniqueness) and assuming that the conductivity rate is positive and the thermal dissipation condition hold, but the elasticity tensor fails to be positive (for the instability). In Section 5, the uniqueness and instability is proved for the same system under an alternative set of assumptions. Section 6 is devoted to the displacement of MGT type in thermoviscoelastic system, and Section 7 to the displacement of MGT type in thermoviscoelastic system of type III, proving also uniqueness and instability under similar assumptions. The techniques used in the previous sections are the logarithmic convexity argument and the Lagrange identities method. To our knowledge, this paper is the first time where these methods are being used for a third order in time system.

Remark 1.1. In all the problems that we consider in this paper, we will be assuming existence in order to prove uniqueness and instability of solutions. Nevertheless, it has to be said that, depending on the initial and boundary conditions, there is no guarantee that such a solution. Hence, we should treat each case carefully.

2. MGT-heat equation: uniqueness and instability

In this section we consider the heat conduction of MGT type problem (see references [5] and [26]) determined by the equations

$$\tau c(\mathbf{x})\dddot{\theta} + c(\mathbf{x})\ddot{\theta} = (k_{ij}(\mathbf{x})\dot{\theta}_{,j})_{,i} + (k^*_{ij}(\mathbf{x})\theta_{,j})_{,i} \tag{2.1}$$

where $\theta(\mathbf{x}, t)$ stands for the temperature, with null Dirichlet boundary conditions

$$\theta(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial B, \ \ t > 0 \tag{2.2}$$

and the initial conditions

$$\theta(\mathbf{x}, 0) = \theta^0(\mathbf{x}), \quad \dot{\theta}(\mathbf{x}, 0) = \vartheta^0(\mathbf{x}), \quad \ddot{\theta}(\mathbf{x}, 0) = \eta^0(\mathbf{x}), \quad \mathbf{x} \in B. \tag{2.3}$$

We recall that we write $\theta_{,i}$ to denote the derivative of the function $\theta(\mathbf{x}, t)$ with respect to the space variable x_i, and $\dot{\theta}$ to denote its derivative with respect to the time variable t.

Apart from the hypotheses considered in Section 1, in the present section we also assume that there exists a positive constant K_0 such that

$$K_{ij}\xi_i\xi_j \geq K_0\xi_i\xi_i, \tag{2.4}$$

for every vector (ξ_i), where $K_{ij} = k_{ij} - \tau k^*_{ij}$.

Also, we assume the thermal capacity $c(\mathbf{x})$ fulfils that

$$c(\mathbf{x}) \geq c_0 > 0, \ \ \mathbf{x} \in B, \tag{2.5}$$

which is a natural condition to consider from the physical point of view.

Observe that we do not impose any condition on the sign of the tensors k_{ij} and k^*_{ij} in the sense that they could be even negative[2] and this fact could be compatible with the condition on the tensor K_{ij}.

We are going to use the logarithmic convexity argument to prove the results of this section (see, for instance, Chpt.4 of [9]), which are uniqueness and instability of the solutions of the problem above. As we said, it is worth noting that this is the first time we see this argument applied to a third order in time equation.

The analysis starts by considering the energy equation

$$E_1(t) = E_1(0) \tag{2.6}$$

where

$$E_1(t) = \int_B (c(\dot{\theta} + \tau\ddot{\theta})^2 + k^*_{ij}(\theta_{,i} + \tau\dot{\theta}_{,i})(\theta_{,j} + \tau\dot{\theta}_{,j}) + \tau K_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j})dv + 2\int_0^t \int_B K_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j}dvds.$$

(notice that the first three terms also depend on t). This energy equation (2.6) can be seen after multiplying equation (2.1) by $\dot{\theta} + \tau\ddot{\theta}$, integrating by parts and using the boundary conditions.

Logarithmic convexity argument is based in the choice of a suitable function defined on the solutions satisfying certain conditions, that we are going to see. In our case the function is:

$$F(t, \omega, t_0) = \int_B c(\theta + \tau\dot{\theta})^2 dv + \int_0^t \int_B K_{ij}\theta_{,i}\theta_{,j}dvds + \omega(t + t_0)^2.$$

Observe that under the assumptions (2.5) and (2.4), F is a strictly positive function.

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[2]However it is worth recalling that the basic axioms of the thermomechanics imply that the tensor k_{ij} is semi-definite positive.

Here ω and t_0 are two positive parameters to be selected later. We have

$$\frac{dF(t,\omega,t_0)}{dt}=2\int_B c(\theta+\tau\dot{\theta})(\dot{\theta}+\tau\ddot{\theta})dv+2\omega(t+t_0)+2\int_0^t\int_B K_{ij}\theta_{,i}\dot{\theta}_{,j}dvds+\int_B K_{ij}\theta_{,i}(0)\theta_{,j}(0)dv.$$

where the last two terms have been obtained after deriving again and then integrating with respect to t, and

$$\frac{d^2F(t,\omega,t_0)}{dt^2}=2\int_B\left(c(\theta+\tau\dot{\theta})(\ddot{\theta}+\tau\dddot{\theta})+c(\dot{\theta}+\tau\ddot{\theta})^2\right)dv+2\int_B K_{ij}\theta_{,i}\dot{\theta}_{,j}dv+2\omega. \tag{2.7}$$

We note that

$$\int_B c(\theta+\tau\dot{\theta})(\ddot{\theta}+\tau\dddot{\theta})dv+\int_B K_{ij}\theta_{,i}\dot{\theta}_{,j}dv=-\int_B(k^*_{ij}(\theta_{,i}+\tau\dot{\theta}_{,i})(\theta_{,j}+\tau\dot{\theta}_{,j})+\tau K_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j})dv. \tag{2.8}$$

where we have used equation (2.1) and integrated by parts.

We use (2.8) and (2.6) in the expression (2.7) to obtain that

$$\frac{d^2F(t,\omega,t_0)}{dt^2}=4\int_B c(\dot{\theta}+\tau\ddot{\theta})^2dv+4\int_0^t\int_B K_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j}dv+2(\omega-E_1(0)).$$

We now obtain

$$\frac{d^2F(t,\omega,t_0)}{dt^2}F(t,\omega,t_0)-\left(\frac{dF(t,\omega,t_0)}{dt}-\frac{\nu}{2}\right)^2\geq-2(\omega+E_1(0))F(t,\omega,t_0), \tag{2.9}$$

where

$$\nu=2\int_B K_{ij}\theta^0_{,i}\theta^0_{,j}dv$$

It is worth noting that if we consider the functional $F(t)=F(t,0,0)$ in the case of null initial conditions, (in which case $\nu=0$ and $E_1(0)=0$) we obtain

$$\frac{d^2F(t)}{dt^2}F(t)-\left(\frac{dF(t)}{dt}\right)^2\geq 0.$$

This is equivalent to

$$\frac{d^2\ln F(t)}{dt^2}\geq 0,$$

that is, $F(t)$ is a logarithmic convex function. From the previous inequality and following the same argument as in Chpt. 4.3.2 of [9] we derive that

$$F(t)\leq F(0)^{1-t/t_1}F(t_1)^{t/t_1},\quad 0\leq t\leq t_1,$$

and we conclude that $F(t)=0,\ 0\leq t\leq t_1$ whenever we assume null initial conditions. From this relation we conclude $u\equiv 0$ if the initial conditions are zero and, hence, the uniqueness of the solution.

In the general case and assuming that $E_1(0)<0$, we can always take $\omega=-E_1(0)$. From (2.9) we obtain that

$$\frac{d^2F(t,\omega,t_0)}{dt^2}F(t,\omega,t_0)\geq\left(\frac{dF(t,\omega,t_0)}{dt}-\frac{\nu}{2}\right)^2\geq\frac{dF(t,\omega,t_0)}{dt}\left(\frac{dF(t,\omega,t_0)}{dt}-\nu\right).$$

Considering t_0 large enough such that $\dot{F}(0,\omega,t_0)>\nu$, the previous inequality implies that

$$\ln\left(\frac{\frac{dF}{dt}(t,\omega,t_0)-\nu}{F(t,\omega,t_0)}\right)\geq\ln\left(\frac{\frac{dF}{dt}(0,\omega,t_0)-\nu}{F(0,\omega,t_0)}\right)$$

which implies

$$\frac{dF}{dt}(t,\omega,t_0) \geq \frac{\frac{dF}{dt}(0,\omega,t_0)-\nu}{F(0,\omega,t_0)}F(t,\omega,t_0)+\nu. \tag{2.10}$$

After integrating (2.10) we obtain

$$F(t,\omega,t_0) \geq \frac{F(0,\omega,t_0)\dot{F}(0,\omega,t_0)}{\dot{F}(0,\omega,t_0)-\nu}\exp\left(\frac{\dot{F}(0,\omega,t_0)-\nu}{F(0,\omega,t_0)}t\right) - \frac{\nu F(0,\omega,t_0)}{\dot{F}(0,\omega,t_0)-\nu}.$$

This inequality gives the exponential growth of the solutions. We have proved that:

Theorem 2.1. *Assume that the symmetry condition* (1.2) *and the positivity assumptions* (2.4) *and* (2.5) *hold, and consider the MGT-heat equation problem* (2.1) *with* (2.2) *and* (2.3) *as boundary and initial conditions. Then*

(i) *this first initial-boundary-value problem* (2.1)-(2.3) *has at most one solution.*
(ii) *If $E_1(0) < 0$, then the solution of this problem becomes unbounded in an exponential way.*

3. MGT thermoelasticity: Uniqueness

The system (usually) called as thermoelasticity of Moore-Gibson-Thompson type (see [5], for instance) is given by

$$\rho\ddot{u}_i = (C^*_{ijkl}u_{k,l} - \beta_{ij}(\dot{\theta}+\tau\ddot{\theta}))_{,j} \tag{3.1}$$

$$\tau c(\mathbf{x})\dddot{\theta} + c(\mathbf{x})\ddot{\theta} = -\beta_{ij}\dot{u}_{i,j} + (k_{ij}(\mathbf{x})\dot{\theta}_{,j})_{,i} + (k^*_{ij}(\mathbf{x})\theta_{,j})_{,i}, \tag{3.2}$$

where we have assumed that the reference temperature is one to simplify the calculations. In this system the vector (u_i) denotes the velocity, $C^*_{ijkl} = C^*_{ijkl}(\mathbf{x})$ is the elasticity tensor and $\beta_{ij} = \beta_{ij}(\mathbf{x})$ is the coupling tensor. We want to state a uniqueness result for the solutions of the problem determined by this system with boundary conditions

$$\theta(\mathbf{x},t) = 0 \text{ and } u_i(\mathbf{x},t) = 0, \quad \mathbf{x}\in\partial B, \; t>0, \tag{3.3}$$

and initial conditions

$$\theta(\mathbf{x},0) = \theta^0(\mathbf{x}), \quad \dot{\theta}(\mathbf{x},0) = \vartheta^0(\mathbf{x}), \quad \ddot{\theta}(\mathbf{x},0) = \eta^0(\mathbf{x}), \quad \mathbf{x}\in B \tag{3.4}$$

and

$$u_i(\mathbf{x},0) = u_i^0(\mathbf{x}), \; \dot{u}_i(\mathbf{x},0) = v_i^0(\mathbf{x}), \quad \mathbf{x}\in B. \tag{3.5}$$

In this section, we do not impose the positivity conditions on the constitutive tensors given in the previous one, but we have to impose another condition on the thermal tensor in order to obtain our results. That is, we assume that there exists a positive constant k_0 such that

$$k_{ij}\xi_i\xi_j \geq k_0\xi_i\xi_i \tag{3.6}$$

for every vector (ξ_i).

With respect ot the mechanical part we also assume the following lower bound for the mass density $\rho(\mathbf{x})$:

$$\rho(\mathbf{x}) \geq \rho_0 > 0, \quad \mathbf{x}\in B, \tag{3.7}$$

which is an obvious assumption, and the symmetry of the elasticity tensor, that is

$$C^*_{ijkl} = C^*_{klij}, \tag{3.8}$$

which is an assumption coming from the axioms of thermoelasticity.

The assumption that the thermal capacity $c(\mathbf{x})$ is strictly positive is not needed.

The goal of this section is to prove the uniqueness of solutions of the problem above. To do so, our argument is based in the Lagrange identities method (see references [9] or [1], for instance).

As the problem is linear, in order to prove the uniqueness of the solutions it is enough to prove that the only solution with null initial conditions is the null solution. Therefore in this section we assume that

$$u_i^0(\mathbf{x}) = v_i^0(\mathbf{x}) = \theta^0(\mathbf{x}) = \vartheta^0(\mathbf{x}) = \eta^0(\mathbf{x}) = 0.$$

In this case of null initial conditions, the energy equation writes

$$E_2(t) = 0 \tag{3.9}$$

with

$$E_2(t) = \int_B (\rho \dot{u}_i \dot{u}_i + C^*_{ijkl} u_{i,j} u_{k,l}) dv + \int_B (c(\dot{\theta} + \tau\ddot{\theta})^2 + k^*_{ij}(\theta_{,i} + \tau\dot{\theta}_{,i})(\theta_{,j} + \tau\dot{\theta}_{,j}) + \tau K_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j}) dv$$
$$+ 2\int_0^t \int_B K_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j} dv ds.$$

This energy equation can be obtained after multiplying equation (3.1) by $\dot{u}_i$ and equation (3.2) by $\dot{\theta} + \tau\ddot{\theta}$, integrating by parts and imposing the null boundary conditions.

We now proceed with the Lagrange identities method, that is, for a fixed $t \in (0, T^*)$ (where $T^* > 0$) we compute the following identities. Denoting $\langle \cdot, \cdot \rangle$ as the usual L^2 product in B, we first multipliy equation (3.1) by $\dot{u}_i(2t - s)$, integrate by parts in the spatial variable and impose the null boundary conditions, obtaining that

(3.10)

$$\int_0^t \langle \rho \ddot{u}_i(s), \dot{u}_i(2t-s) \rangle ds + \int_0^t \langle C^*_{ijkl} u_{k,l}(s), \dot{u}_{i,j}(2t-s) \rangle ds = \int_0^t \langle \beta_{ij}(\dot{\theta}(s) + \tau\ddot{\theta}(s)), \dot{u}_{i,j}(2t-s) \rangle ds,$$

and, analogously, we obtain that

$$\int_0^t \langle \rho \ddot{u}_i(2t-s), \dot{u}_i(s) \rangle ds + \int_0^t \langle C^*_{ijkl} u_{k,l}(2t-s), \dot{u}_{i,j}(s) \rangle ds$$
$$= \int_0^t \langle \beta_{ij}\left(\dot{\theta}(2t-s) + \tau\ddot{\theta}(2t-s)\right), \dot{u}_{i,j}(s) \rangle ds, \tag{3.11}$$

Now, multiplying equation (3.2) by $\dot{\theta}(2t-s) + \tau\ddot{\theta}(2t-s)$, integrating by parts and imposing the null boundary conditions we obtain

(3.12)

$$\int_0^t \langle c\left(\ddot{\theta}(s) + \tau\dddot{\theta}(s)\right), \left(\dot{\theta}(2t-s) + \tau\ddot{\theta}(2t-s)\right) \rangle ds + \int_0^t \langle k^*_{ij}\theta_{,i}(s), \left(\dot{\theta}_{,j}(2t-s) + \tau\ddot{\theta}_{,j}(2t-s)\right) \rangle ds$$
$$+ \int_0^t \langle k_{ij}\dot{\theta}_{,i}(s), \left(\dot{\theta}_{,j}(2t-s) + \tau\ddot{\theta}_{,j}(2t-s)\right) \rangle ds = -\int_0^t \langle \beta_{ij}\dot{u}_{i,j}(s), \left(\dot{\theta}(2t-s) + \tau\ddot{\theta}(2t-s)\right) \rangle ds,$$

and, similarly,

$$\int_0^t \langle c\left(\ddot{\theta}(2t-s)+\tau\dddot{\theta}(2t-s)\right), \dot{\theta}(s)+\tau\ddot{\theta}(s)\rangle ds+\int_0^t \langle k^*_{ij}\theta_{,i}(2t-s), \dot{\theta}_{,j}(s)+\tau\ddot{\theta}_{,j}(s)\rangle ds$$
$$+\int_0^t \langle k_{ij}\dot{\theta}_{,i}(2t-s), \dot{\theta}_{,j}(s)+\tau\ddot{\theta}_{,j}(s)\rangle ds=-\int_0^t \langle \beta_{ij}\dot{u}_{i,j}(2t-s), \dot{\theta}(s)+\tau\ddot{\theta}(s)\rangle ds. \tag{3.13}$$

Now, we form the combination (3.10)+(3.13)-(3.11)-(3.12). First, (3.10)+(3.13) cancels the right hand side of both equalities, and the same happens with (3.11)+(3.12). Then, combining them as (3.10)+(3.13)-(3.11)-(3.12), after integration over s, considering the symmetry of the operators, and taking into account that we are assuming null initial conditions, we obtain (everything evaluated in t)

$$\int_B (\rho\dot{u}_i\dot{u}_i+k^*_{ij}\theta_{,i}\theta_{,j}+\tau k_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j}+2\tau k^*_{ij}\theta_{,i}\dot{\theta}_{,j})dv=\int_B (C^*_{ijkl}u_{i,j}u_{k,l}+c(\dot{\theta}+\tau\ddot{\theta})^2)dv.$$

Substituting the previous equality into the energy equation (3.9) we obtain

$$\int_B (\rho\dot{u}_i\dot{u}_i+k^*_{ij}\theta_{,i}\theta_{,j}+\tau k_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j}+2\tau k^*_{ij}\theta_{,i}\dot{\theta}_{,j})dv+\int_0^t\int_B K_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j}dvds=0.$$

After time integration, and using again the null initial conditions, we also have

$$\int_0^t\int_B (\rho\dot{u}_i\dot{u}_i+k^*_{ij}\theta_{,i}\theta_{,j}+\tau k_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j}+2\tau k^*_{ij}\theta_{,i}\dot{\theta}_{,j})dvds+\int_0^t\int_B (t-s)K_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j}dvds=0. \tag{3.14}$$

Let us consider

$$I_1=\int_0^t\int_B k^*_{ij}\theta_{,i}\theta_{,j}dvds \quad \text{and} \quad I_2=2\tau\int_0^t\int_B k^*_{ij}\theta_{,i}\dot{\theta}_{,j}dvds.$$

As we are assuming null initial conditions we have

$$|I_1|\leq K_1\left(\int_0^t\int_B \theta_{,i}\theta_{,i}dvds\right)\leq K_2t^2\int_0^t\int_B k_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j}dvds \tag{3.15}$$

Here K_1 and K_2 are computable positive constants, and we have used the Poincaré inequality taking into account that the solutions vanish at $t=0$ and the positivity of k_{ij} assumed in (3.6). Similarly, using the same tools and assumptions,

$$|I_2|=|2\tau\int_0^t\int_B k^*_{ij}\theta_{,i}\dot{\theta}_{,j}dvds|\leq K_3\int_0^t\int_B|\theta_{,i}\dot{\theta}_{,i}|dvds \tag{3.16}$$

$$\leq K_3\left(\int_0^t\int_B \theta_{,i}\theta_{,i}dvds\right)^{1/2}\left(\int_0^t\int_B \dot{\theta}_{,i}\dot{\theta}_{,i}dvds\right)^{1/2}\leq K_4t\int_0^t\int_B k_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j}dvds$$

where K_3 and K_4 are also computable positive constants.

We select t_1 small enough to guarantee that $\tau-K_2t_1^2-K_4t_1>\tau/2$, but positive. Using this and inequalities (3.15) and (3.16), it is easy to see that for every $t\leq t_1$ the following inequality

$$\int_0^t\int_B k^*_{ij}(\theta_{,i}\theta_{,j}+2\tau\theta_{,i}\dot{\theta}_{,j})dvds+\tau\int_0^t\int_B k_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j}dvds\geq\frac{\tau}{2}\int_0^t\int_B k_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j}dvds \tag{3.17}$$

is satisfied.

Now, we define the function

$$G(t) = -\int_0^t \int_B (t-s) K_{ij} \dot{\theta}_{,i} \dot{\theta}_{,j} dv ds$$

and observe that, by (3.14),

$$G(t) = \int_0^t \int_B \rho \dot{u}_i \dot{u}_i dv ds + \int_0^t \int_B k^*_{ij} (\theta_{,i}\theta_{,j} + 2\tau \theta_{,i} \dot{\theta}_{,j}) dv ds + \tau \int_0^t \int_B k_{ij} \dot{\theta}_{,i} \dot{\theta}_{,j} dv ds. \tag{3.18}$$

We have

$$\dot{G}(t) = -\int_0^t \int_B K_{ij} \dot{\theta}_{,i} \dot{\theta}_{,j} dv ds \leq K_5 G(t) \quad t \in [0, t_1],$$

where K_5 can be calculated. Observe that, by (3.17) and (3.18), we have $G(t) \geq 0$ for $t \in [0, t_1]$. After integration we see

$$G(t) \leq G(0) \exp(K_5 t) \quad t \in [0, t_1].$$

In view of (3.17) and (3.18) we have

$$\int_0^t \int_B (\rho \dot{u}_i \dot{u}_i + \frac{\tau}{2} k_{ij} \dot{\theta}_{,i} \dot{\theta}_{,j}) dv ds \leq G(0) \exp(K_5 t) \quad t \in [0, t_1].$$

As $G(0) = 0$, it then follows that $u_i = \theta = 0$ when $t \in [0, t_1]$. We can extend this argument to the interval $[t_1, 2t_1]$, $[2t_1, 3t_1]$,... to see that the solution vanishes for every time.

Therefore we have proved:

Theorem 3.1. *Assume that the symmetry conditions* (1.2) *and* (3.8)*, and the positivity assumptions* (3.6) *and* (3.7) *hold. Then the first initial-boundary-value problem* (3.1)-(3.5) *has at most one solution.*

Remark 3.2. It is important to note that to obtain this result we have not imposed any condition on the dissipation term K_{ij}.

4. MGT thermoelasticity: Instability

The aim of this section is to obtain an instability result for the solutions of the problem determined by the system (3.1)-(3.2) with the homogeneous Dirichlet boundary conditions (3.3) and the general initial conditions (3.4)-(3.5) given in the previous section. In this section we continue assuming that the mass density $\rho(\mathbf{x})$ is definite positive and the symmetry of the elasticity tensor C^*_{ijkl} as in the previous section (see (3.7) and (3.8)), but not the positivity of k_{ij}. Instead, we need to impose positivity conditions on the tensors k^*_{ij} and K_{ij} (which, in turn, imply the positivity of k_{ij}). These are the requirements needed in order to use again the logarithmic convexity method, that will allow us to prove instability results for this problem under this more restrictive set of hypothesis.

So, and as in Section 2, we assume the positivity of K_{ij} (see (2.4)) as well as of k^*_{ij}, that is, we assume the existence of another positive constant k^*_0 such that

$$k^*_{ij} \xi_i \xi_j \geq k^*_0 \xi_i \xi_i, \tag{4.1}$$

for every vector (ξ_i). As we said above, it is worth noting that previous inequalities imply that the tensor k_{ij} is also positive definite (that is, hypothesis (3.6)).

If we integrate with respect to the time t the heat equation (3.2) we obtain

$$c\dot{\theta} + c\tau\ddot{\theta} = -\beta_{ij}u_{i,j} + (k_{ij}\theta_{,i})_{,j} + (k^*_{ij}\alpha_{,i})_{,j} + c\vartheta^0 + c\tau\eta^0 + \beta_{ij}u^0_{i,j} - (k_{ij}\theta^0_{,i})_{,j},$$

where

$$\alpha(\mathbf{x}, t) = \int_0^t \theta(x, s)ds.$$

If we denote by $\chi(\mathbf{x})$ the solution of the problem

$$(k^*_{ij}\chi_{,i})_{,j} = c\vartheta^0 + c\tau\eta^0 + \beta_{ij}u^0_{i,j} - (k_{ij}\theta^0_{,i})_{,j},$$

with homogeneous Dirichlet boundary conditions, we can write

$$c\dot{\theta} + c\tau\ddot{\theta} = -\beta_{ij}u_{i,j} + (k_{ij}\theta_{,i})_{,j} + (k^*_{ij}\phi_{,i})_{,j}, \tag{4.2}$$

where

$$\phi(\mathbf{x}, t) = \alpha(\mathbf{x}, t) + \chi(\mathbf{x}). \tag{4.3}$$

The analysis starts by considering again the energy equation

$$E_2(t) = E_2(0) \tag{4.4}$$

where, as in the previous section,

$$\begin{aligned} E_2(t) = &\int_B (\rho\dot{u}_i\dot{u}_i + C^*_{ijkl}u_{i,j}u_{k,l})dv + \int_B (c(\dot{\theta} + \tau\ddot{\theta})^2 + k^*_{ij}(\theta_{,i} + \tau\dot{\theta}_{,i})(\theta_{,j} + \tau\dot{\theta}_{,j}) + \tau K_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j})dv \\ &+ 2\int_0^t \int_B K_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j}dvds. \end{aligned}$$

We recall that, as we are not assuming now null initial conditions, $E_2(0)$ may not be equal to zero, as happened in (3.9).

Now, we define the function

$$F(t, \omega, t_0) = \int_B (\rho u_i u_i + k^*_{ij}(\phi_{,i} + \tau\theta_{,i})(\phi_{,j} + \tau\theta_{,j}) + \tau K_{ij}\theta_{,i}\theta_{,j})dv + \int_0^t \int_B K_{ij}\theta_{,i}\theta_{,j}dvds + \omega(t + t_0)^2.$$

Observe that under the assumptions assumed in this section F is a strictly positive function.

Proceeding in the same way as in Section 2, we have

$$\begin{aligned} \frac{dF(t, \omega, t_0)}{dt} = &\, 2\int_B (\rho u_i\dot{u}_i + k^*_{ij}(\phi_{,i} + \tau\theta_{,i})(\theta_{,j} + \tau\dot{\theta}_{,j}) + \tau K_{ij}\theta_{,i}\dot{\theta}_{,j})dv \\ &+ 2\int_0^t \int_B K_{ij}\theta_{,i}\dot{\theta}_{,j}dvds + \int_B K_{ij}\theta^0_{,i}\theta^0_{,j}dv + 2\omega(t + t_0). \end{aligned}$$

where $\phi(\mathbf{x}, t)$ is defined in (4.3), and we also have

$$\begin{aligned} \frac{d^2F(t, \omega, t_0)}{dt^2} = &\, 2\int_B (\rho u_i\ddot{u}_i + k^*_{ij}(\phi_{,i} + \tau\theta_{,i})(\dot{\theta}_{,j} + \tau\ddot{\theta}_{,j}) + \tau K_{ij}\theta_{,i}\ddot{\theta}_{,j})dv + 2\int_B K_{ij}\theta_{,i}\dot{\theta}_{,j}dv \\ &+ 2\omega + 2\int_B (\rho\dot{u}_i\dot{u}_i + k^*_{ij}(\theta_{,i} + \tau\dot{\theta}_{,i})(\theta_{,j} + \tau\dot{\theta}_{,j}) + \tau K_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j})dv. \end{aligned}$$

Using equation (3.1) multiplied by u_i and integrated by parts, and equation (4.2) multiplied by $\dot{\theta} + \tau\ddot{\theta}$ and also integrated by parts (but only the second term of its right hand side), we note that

$$\frac{d^2F(t,\omega,t_0)}{dt^2} = -2\int_B (C^*_{ijkl}u_{i,j}u_{k,l} + c(\dot{\theta}+\tau\ddot{\theta})^2)dv$$
$$+ 2\omega + 2\int_B (\rho\dot{u}_i\dot{u}_i + k^*_{ij}(\theta_{,i}+\tau\dot{\theta}_{,i})(\theta_{,j}+\tau\dot{\theta}_{,j}) + \tau K_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j})dv.$$

Recalling the definition of $E_2(t)$ and using the energy equation (4.4) we have

$$\frac{d^2F(t,\omega,t_0)}{dt^2} = 4\int_B (\rho\dot{u}_i\dot{u}_i + k^*_{ij}(\theta_{,i}+\tau\dot{\theta}_{,i})(\theta_{,j}+\tau\dot{\theta}_{,j}) + \tau K_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j})dv + 4\int_0^t\int_B K_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j}dvds + 2(\omega - E_2(0)).$$

We obtain again

$$\frac{d^2F(t,\omega,t_0)}{dt^2}F(t,\omega,t_0) - \left(\frac{dF(t,\omega,t_0)}{dt} - \frac{\nu}{2}\right)^2 \geq -2(\omega + E_2(0))F(t,\omega,t_0). \tag{4.5}$$

Here

$$\nu = 2\int_B K_{ij}\theta^0_{,i}\theta^0_{,j}dv.$$

In the case that $E_2(0) < 0$, we can always take $\omega = -E_2(0)$ and t_0 large enough to guarantee that $\dot{F}(0,\omega,t_0) > \nu$, and then, from (4.5) and proceeding in the same way as in Section 2, we obtain that

$$F(t,\omega,t_0) \geq \frac{F(0,\omega,t_0)\dot{F}(0,\omega,t_0)}{\dot{F}(0,\omega,t_0) - \nu}\exp\left(\frac{\dot{F}(0,\omega,t_0) - \nu}{F(0,\omega,t_0)}t\right) - \frac{\nu F(0,\omega,t_0)}{\dot{F}(0,\omega,t_0) - \nu} \tag{4.6}$$

which gives us the exponential growth of the solutions.

Hence, we have proved the following result:

Theorem 4.1. *Assume that the symmetry conditions* (1.2) *and* (3.8)*, and the positivity assumptions* (2.4), (3.7)*, and* (4.1) *hold. Then, if* $E_2(0) < 0$ *the solution of the initial-boundary-value problem* (3.1)-(3.5) *becomes unbounded in an exponential way.*

Remark 4.2. Observe that this logarithmic convexity technique would also us to prove the uniqueness of solutions of problem (3.1)-(3.5) under the hypotheses considered in Theorem 4.1. However, we recall that this uniqueness has already been proved in Section 3 under a less restrictive set of hypotheses.

Remark 4.3. Inequality (4.6) also states that the solutions of our system are not stable, in the sense that small changes in the initial state of the system become larger as time increases. Physically, that means that for prestressed elastic solids, the MGT-thermal effect may not control the instability given by the lack of positivity of the elastic tensor. Of course, in case of positivity of the elasticity tensor, the solutions are stable. Furthermore, we have seen that the thermal effects proposed here are not enough to stabilize the elastic deformations.

5. MGT thermoelasticity: Another approach

The aim of this section is to obtain another result about uniqueness and instability for the solutions of the problem of the MGT-thermoelasticty (3.1)-(3.5) under an alternative family of assumptions. In this section we assume the conditions proposed in the previous section about the thermal constitutive tensors in the sense that c and K_{ij} are positive (see conditions (2.4) and (2.5)) but we do not assume the positivity of k_{ij} neither of k^*_{ij}. We also assume that there exists a positive constant C^* such that

$$\int_B C^*_{ijkl} u_{i,j} u_{k,l} dv \geq C^* \int_B u_{i,j} u_{i,j} dv, \text{ for all } (u_i) \text{ such that } u_i|_{\partial B} = 0. \tag{5.1}$$

In this situation, we can use again the logarithmic convexity method to prove uniqueness and instability of solutions. To do so, it is more convenient to consider the new alternative energy

$$\begin{aligned} E^*_2(t) = &\int_B (\rho \ddot{u}_i \ddot{u}_i + C^*_{ijkl} \dot{u}_{i,j} \dot{u}_{k,l} + c(\ddot{\theta} + \tau \dddot{\theta})^2 + k^*_{ij}(\dot{\theta}_{,i} + \tau \ddot{\theta}_{,i})(\dot{\theta}_{,j} + \tau \ddot{\theta}_{,j}) + \tau K_{ij} \ddot{\theta}_{,i} \ddot{\theta}_{,j}) dv \\ &+ 2 \int_0^t \int_B K_{ij} \ddot{\theta}_{,i} \ddot{\theta}_{,j} dv ds. \end{aligned} \tag{5.2}$$

Using the temporal derivatives of system (3.1)-(3.2), it is easy to see that this new energy is also conserved, and, hence

$$E^*_2(t) = E^*_2(0). \tag{5.3}$$

We now define the function

$$F(t, \omega, t_0) = \int_B \left(c(\dot{\theta} + \tau \ddot{\theta})^2 + C^*_{ijkl} u_{i,j} u_{k,l} \right) dv + \int_0^t \int_B K_{ij} \dot{\theta}_{,i} \dot{\theta}_{,j} dv ds + \omega (t + t_0)^2. \tag{5.4}$$

Observe that under the assumptions assumed in this section F is a strictly positive function.

Proceeding as in Sections 2 or 4, we obtain again an inequality such as (2.9). We start by deriving F with respect to time, and we obtain

$$\begin{aligned} \frac{dF(t, \omega, t_0)}{dt} = &2 \int_B \left(c(\dot{\theta} + \tau \ddot{\theta})(\ddot{\theta} + \tau \dddot{\theta}) + C^*_{ijkl} u_{i,j} \dot{u}_{k,l} \right) dv \\ &+ 2 \int_0^t \int_B K_{ij} \dot{\theta}_{,i} \ddot{\theta}_{,j} dv ds + \int_B K_{ij} \vartheta^0_{,i} \vartheta^0_{,j} dv + 2\omega (t + t_0) \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} \frac{d^2F(t, \omega, t_0)}{dt^2} = &2 \int_B \left(c(\ddot{\theta} + \tau \dddot{\theta})^2 + c(\dot{\theta} + \tau \ddot{\theta})(\dddot{\theta} + \tau \ddddot{\theta}) \right) dv \\ &+ 2 \int_B \left(C^*_{ijkl} \dot{u}_{i,j} \dot{u}_{k,l} + C^*_{ijkl} u_{i,j} \ddot{u}_{k,l} \right) dv + 2 \int_B K_{ij} \dot{\theta}_{,i} \ddot{\theta}_{,j} dv + 2\omega. \end{aligned}$$

We now multiply (3.1) by $\ddot{u}_i$ and (3.2) by $\dot{\theta} + \tau \ddot{\theta}$, integrate each one on B, and use the resulting equalities into the previous expression, obtaining:

$$\begin{aligned} \frac{d^2F(t, \omega, t_0)}{dt^2} = &2 \int_B c(\ddot{\theta} + \tau \dddot{\theta})^2 dv - 2 \int_B \left(k_{ij} \ddot{\theta}_j (\dot{\theta}_{,i} + \tau \ddot{\theta}_{,i}) + k^*_{ij} \dot{\theta}_{,j} (\dot{\theta}_{,i} + \tau \ddot{\theta}_{,i}) \right) dv \\ &+ 2 \int_B C^*_{ijkl} \dot{u}_{i,j} \dot{u}_{k,l} dv - 2 \int_B \rho (\ddot{u}_i)^2 dv + 2 \int_B K_{ij} \dot{\theta}_{,i} \ddot{\theta}_{,j} dv + 2w. \end{aligned}$$

Substituting the energy equation (5.2) into the previous equality, we obtain the following alternative expression for $\frac{d^2F}{dt^2}$:

(5.6)

$$\frac{d^2F(t,\omega,t_0)}{dt^2} = 4\int_B c(\ddot{\theta}+\tau\dddot{\theta})^2 dv + 4\int_B C^*_{ijkl}\dot{u}_{i,j}\dot{u}_{k,l}dv + 4\int_0^t\int_B K_{ij}\ddot{\theta}_{,i}\ddot{\theta}_{,j}dvds + 2(\omega - E^*_2(0)).$$

Using expressions (5.4), (5.5) and (5.6), we can see that inequality (2.9) is satisfied with

$$\nu = 2\int_B K_{ij}\vartheta^0_{,i}\vartheta^0_{,j}dv.$$

Therefore we have proved the following result.

Theorem 5.1. *Assume that the symmetry condition* (1.2)*, and the positivity assumptions* (2.4)*,* (2.5) *and* (5.1) *hold. Then,*

(i) *The first initial-boundary-value problem* (3.1)-(3.5) *has at most one solution.*
(ii) *If $E^*_2(0) < 0$, then the solution of* (3.1)-(3.5) *becomes unbounded in an exponential way.*

Remark 5.2. It is worth noting that even in case that the elasticity tensor C^*_{ij} must be positive definite, the initial energy of the system could be negative as we are not assuming the positivity of the tensors k_{ij} neither k^*_{ij}. Hence, together with hypothesis in Section 4, we see another possibility for the instability of the solutions.

6. MGT for the displacement in thermoviscoelasticity

In this section we consider the problem determined by a viscoelastic material coupled with the Fourier thermal effects. If we consider a relaxation function

$$G_{ijkl}(\mathbf{x},s) = C^*_{ijkl}(\mathbf{x}) + \exp(-\tau^{-1}s)(\tau^{-1}C_{ijkl}(\mathbf{x}) - C^*_{ijkl}(\mathbf{x}))$$

and coupling the equation for the displacement with the usual Fourier heat equation we obtain the system

$$\tau\rho\dddot{u}_i + \rho\ddot{u}_i = (C^*_{ijkl}u_{k,l} + C_{ijkl}\dot{u}_{k,l} - \beta_{ij}\theta)_{,j}, \tag{6.1}$$

$$c\dot{\theta} = (k_{ij}\theta_{,i})_{,j} - \beta_{ij}\dot{u}_{i,j} - \tau\beta_{ij}\ddot{u}_{i,j}. \tag{6.2}$$

We observe that the MGT type form is now on the displacement equation.

We want to state uniqueness and instability results for the solutions of the problem determined by this system with boundary conditions

$$\theta(\mathbf{x},t) = 0 \quad \text{and} \quad u_i(\mathbf{x},t) = 0, \quad \mathbf{x}\in\partial B, \ t > 0, \tag{6.3}$$

and initial conditions

$$u_i(\mathbf{x},0) = u^0_i(\mathbf{x}), \ \dot{u}_i(\mathbf{x},0) = v^0_i(\mathbf{x}), \ \ddot{u}_i(\mathbf{x},0) = w^0_i(\mathbf{x}), \quad \mathbf{x}\in B. \tag{6.4}$$

and

$$\theta(\mathbf{x},0) = \theta^0(\mathbf{x}), \quad \mathbf{x}\in B. \tag{6.5}$$

In this section we assume that the viscoelasticity tensor C_{ijkl}, C^*_{ijkl} and k_{ij} are symmetric in the sense that

$$C_{ijkl} = C_{klij}, \ C^*_{ijkl} = C^*_{klij}, \ k_{ij} = k_{ji}. \tag{6.6}$$

We also assume the positivity of the thermal conductivity tensor k_{ij} given in (3.6). We also impose the the positivity of the tensor $\overline{C}_{ijkl}=C_{ijkl}-\tau C^*_{ijkl}$, that is, the existence of a positive constant $\overline{C}$ such that

$$\int_B \overline{C}_{ijkl}u_{i,j}u_{k,l}dv \geq \overline{C}\int_B u_{i,j}u_{i,j}dv \text{ for all } (u_i) \text{ such that } u_i|_{\partial B}=0. \tag{6.7}$$

We also assume that the mass density $\rho(\mathbf{x})$ is positive, that is (3.7).

If we integrate with respect to the time t the heat equation (6.2) we obtain

$$c\theta=(k_{ij}\alpha_{,i})_{,j}-\beta_{ij}u_{i,j}-\tau\beta_{ij}\dot{u}_{i,j}+c\theta^0+\beta_{i,j}u^0_{i,j}+\tau\beta_{ij}v^0_{i,j}.$$

If we denote by $\chi(\mathbf{x})$ the solution of the problem

$$(k_{ij}\chi_{,i})_{,j}=c\theta^0+\beta_{ij}u^0_{i,j}+\tau\beta_{ij}v^0_{i,j},$$

with homogeneous Dirichlet boundary condition, we can write

$$c\theta=(k_{ij}\phi_{,i})_{,j}-\beta_{i,j}u_{i,j}-\tau\beta_{i,j}\dot{u}_{i,j}. \tag{6.8}$$

where

$$\phi(\mathbf{x},t)=\alpha(\mathbf{x},t)+\chi(\mathbf{x}).$$

The energy equation in this case reads

$$E_3(t)=E_3(0) \tag{6.9}$$

where

$$\begin{aligned}E_3(t)=&\int_B \left(\rho(\dot{u}_i+\tau\ddot{u}_i)(\dot{u}_i+\tau\ddot{u}_i)+C^*_{ijkl}(u_{i,j}+\tau\dot{u}_{i,j})(u_{k,l}+\tau\dot{u}_{k,l})+\tau\overline{C}_{ijkl}\dot{u}_{i,j}\dot{u}_{k,l}+c\theta^2\right)dv\\&+2\int_0^t\int_B\left(\overline{C}_{ijkl}\dot{u}_{i,j}\dot{u}_{k,l}+k_{ij}\theta_{,i}\theta_{,j}\right)dvds.\end{aligned}$$

In this situation we can define the function

$$F(t,\omega,t_0)=\int_B\rho(u_i+\tau\dot{u}_i)(u_i+\tau\dot{u}_i)dv+\int_0^t\int_B\left(\overline{C}_{ijkl}u_{i,j}u_{k,l}+k_{ij}\phi_{,i}\phi_{,j}\right)dvds+\omega(t+t_0)^2. \tag{6.10}$$

Observe that under the assumptions (3.6), (3.7) and (6.7), F is a strictly positive function.

We proceed as in the previous sections and start by deriving F with respect to time:

$$\begin{aligned}\frac{dF(t,\omega,t_0)}{dt}=&2\int_B\rho(u_i+\tau\dot{u}_i)(\dot{u}_i+\tau\ddot{u}_i)dv+2\int_0^t\int_B\left(\overline{C}_{ijkl}\dot{u}_{i,j}u_{k,l}+k_{ij}\theta_{,i}\phi_{,j}\right)dvds\\&+\int_B\left(\overline{C}_{ijkl}u^0_{i,j}u^0_{k,l}+k_{ij}\chi^0_{,i}\chi^0_{,j}\right)dv+2\omega(t+t_0)\end{aligned} \tag{6.11}$$

and

$$\begin{aligned}\frac{d^2F(t,\omega,t_0)}{dt^2}=&2\int_B\left(\rho(\dot{u}_i+\tau\ddot{u}_i)^2+\rho(u_i+\tau\dot{u}_i)(\ddot{u}_i+\tau\dddot{u}_i)\right)dv\\&+2\int_B\left(\overline{C}_{ijkl}\dot{u}_{i,j}u_{k,l}+k_{ij}\theta_{,i}\phi_{,j}\right)dv+2\omega\end{aligned}$$

We now multiply (6.1) by $u_i + \tau \dot{u}_i$ and (6.8) by θ, integrate each one on B, and use the resulting equalities into the previous expression, obtaining:

$$\frac{d^2F(t,\omega,t_0)}{dt^2} = 2\int_B \rho(\dot{u}_i + \tau\ddot{u}_i)^2 dv - 2\int_B c\theta^2 dv$$
$$- 2\int_B \left(C^*_{ijkl}u_{k,l}(u_{i,j} + \tau\dot{u}_{i,j}) + C_{ijkl}\dot{u}_{k,l}(u_{i,j} + \tau\dot{u}_{i,j})\right) dv + 2\int_B \overline{C}_{ijkl}\dot{u}_{i,j}u_{k,l} + 2\omega.$$

Substituting the energy equation (6.9) into the previous equality, we obtain the following alternative expression for $\frac{d^2F}{dt^2}$:

$$\frac{d^2F(t,\omega,t_0)}{dt^2} = 4\int_B \rho(\dot{u}_i + \tau\ddot{u}_i)^2 dv + 4\int_0^t\int_B \left(\overline{C}_{ijkl}\dot{u}_{i,j}\dot{u}_{k,l} + k_{ij}\theta_{,i}\theta_{,j}\right) dvds + 2(\omega - E_3(0)). \tag{6.12}$$

Using expressions (6.10), (6.11) and (6.12), we can see that inequality (2.9) is satisfied with

$$\nu = 2\int_B \left(\overline{C}_{ijkl}u^0_{i,j}u^0_{k,l} + k_{ij}\chi_{,i}\chi_{,j}\right) dv.$$

Therefore we see that:

Theorem 6.1. *Assume that the symmetry conditions* (1.2) *and* (6.6)*, and the positivity assumptions* (3.6)*,* (3.7)*, and* (6.7) *hold. Consider the first initial-boundary-value problem* (6.1)-(6.2) *with the corresponding boundary and initial conditions* (6.3)-(6.5)*. Then,*

(i) *The first initial-boundary-value problem* (6.1)-(6.5) *has at most one solution.*
(ii) *If $E_3(0) < 0$, the solution of* (6.1)-(6.5) *becomes unbounded in an exponential way.*

Remark 6.2. Again we see that the thermal and mechanical dissipation are not so strong to stabilize the mechanical part.

Remark 6.3. It is worth noting that a suitable variation in the logarithmic convexity argument allows us to obtain Holder stability of the solutions in a similar way to the one proposed by Ames and Straughan in [1] for the classical theory of thermoelasticity.

The uniqueness result can be also obtained using the Lagrange identities method. In this case we would need the additional hypothesis of the the positivity of the tensor C_{ijkl}, that is, the existence of a positive constant C such that

$$\int_B C_{ijkl}u_{i,j}u_{k,l}dv \geq C\int_B u_{i,j}u_{i,j}dv \text{ for all } (u_i) \text{ such that } u_i|_{\partial B} = 0. \tag{6.13}$$

If we assume null initial conditions, the Lagrange identities argument used in Section 3 bring us to the relation

$$\int_B \rho(\dot{u}_i + \tau\ddot{u}_i)(\dot{u}_i + \tau\ddot{u}_i)dv = \int_B \left(C^*_{ijkl}u_{i,j}u_{k,l} + 2\tau C^*_{ijkl}\dot{u}_{i,j}u_{k,l} + \tau C_{ijkl}\dot{u}_{i,j}\dot{u}_{k,l} + c\theta^2\right) dv.$$

When considering null initial conditions we have $E_3(0) = 0$. Therefore, substituting the previous equality into the energy equation (6.9) we obtain the relation

$$\int_B \left(c\theta^2 + C^*_{ijkl}u_{i,j}u_{k,l} + 2\tau C^*_{ijkl}\dot{u}_{i,j}u_{k,l} + \tau C_{ijkl}\dot{u}_{i,j}\dot{u}_{k,l}\right) dv + \int_0^t\int_B \left(\overline{C}_{ijkl}\dot{u}_{i,j}\dot{u}_{k,l} + k_{ij}\theta_{,i}\theta_{,j}\right) dvds = 0.$$

After integration by parts and using again the null initial conditions we obtain

$$\begin{aligned}(6.14)\qquad &\int_0^t\int_B\left(c\theta^2+C^*_{ijkl}u_{i,j}u_{k,l}+2\tau C^*_{ijkl}\dot{u}_{i,j}u_{k,l}+\tau C_{ijkl}\dot{u}_{i,j}\dot{u}_{k,l}\right)dvds\\ &+\int_0^t\int_B(t-s)\left(\overline{C}_{ijkl}\dot{u}_{i,j}\dot{u}_{k,l}+k_{ij}\theta_{,i}\theta_{,j}\right)dvds=0.\end{aligned}$$

We now consider

$$J_1=\int_0^t\int_B C^*_{ijkl}u_{i,j}u_{k,l}dvds\quad\text{and}\quad J_2=2\tau\int_0^t\int_B C^*_{ijkl}u_{i,j}\dot{u}_{k,l}dvds.$$

Proceeding as (3.15) and (3.16), and as we are assuming null initial conditions, we have

$$(6.15)\qquad |J_1|\le K_1\left(\int_0^t\int_B u_{i,j}u_{i,j}dvds\right)\le K_2t^2\int_0^t\int_B C_{ijkl}\dot{u}_{i,j}\dot{u}_{k,l}dvds$$

and

$$(6.16)\qquad |J_2|=|2\tau\int_0^t\int_B C^*_{ijkl}u_{i,j}\dot{u}_{k,l}dvds|\le K_3\int_0^t\int_B|u_{i,j}\dot{u}_{k,l}|dvds$$

$$\le K_3\left(\int_0^t\int_B|u_{i,j}|^2dvds\right)^{1/2}\left(\int_0^t\int_B|\dot{u}_{k,l}|^2dvds\right)^{1/2}\le K_4t\int_0^t\int_B C_{ijkl}\dot{u}_{i,j}\dot{u}_{i,j}dvds$$

Here K_i, $i=1,\dots,4$, are computable positive constants, and we have used the Poincaré inequality taking into account that the solutions vanish at $t=0$ and the positivity of C_{ijkl} assumed in (6.13). As in Section 3, we select t_1 small enough to guarantee that $\tau-K_2t_1^2-K_4t_1>\tau/2$, but positive. Using the previous inequalities it is easy to see that for every $t\le t_1$ we have

$$(6.17)\quad \int_0^t\int_B\left(C^*_{ijkl}u_{i,j}u_{k,l}+2\tau C^*_{ijkl}u_{i,j}\dot{u}_{k,l}+\tau C_{ijkl}\dot{u}_{i,j}\dot{u}_{k,l}\right)dvds\ge\frac{\tau}{2}\int_0^t\int_B C_{ijkl}\dot{u}_{i,j}\dot{u}_{k,l}dvds.$$

In case that we assume that the thermal capacity is positive (see (2.5)), the thermal conductivity tensor k_{ij} is positive semi-definite and the tensor C_{ijkl} is positive definite (see (6.13)). Then we can define

$$G(t)=-\int_0^t\int_B(t-s)\left(\overline{C}_{ijkl}\dot{u}_{i,j}\dot{u}_{k,l}+k_{ij}\theta_{,i}\theta_{,j}\right)dvds.$$

By (6.14) we have

$$(6.18)\qquad G(t)=\int_0^t\int_B\left(c\theta^2+C^*_{ijkl}(u_{i,j}u_{k,l}+2\tau u_{i,j}\dot{u}_{k,l})+\tau C_{ijkl}\dot{u}_{i,j}\dot{u}_{k,l}\right)dvds$$

An argument similar to the one proposed in Section 3 but now using (6.17) and (6.18) allows us to obtain again that the only solution is the null solution. Therefore we have obtained an alternative proof for the uniqueness of problem (6.1)-(6.2) when the thermal capacity is positive and we have null initial conditions, under the new assumptions.

Theorem 6.4. *Assume that the symmetry conditions* (1.2) *and* (6.6)*, and the positivity assumptions* (2.5)*,* (3.7) *and* (6.13) *hold. Also assume that the tensor k_{ij} is positive semi-definite. Then, the first initial-boundary-value problem* (6.1)-(6.5) *has at most one solution.*

Remark 6.5. It is important to note that to obtain this second result it is only necessary the positivity of C_{ijkl}, but not of C^*_{ijkl} nor $\overline{C}_{ijkl}$.

7. MGT for the displacement in thermoviscoelasticity of type III

Type III thermoelasticity can be considered if we change the last equation in (6.1)-(6.2), obtaining the system

$$\tau\rho\dddot{u}_i+\rho\ddot{u}_i=(C^*_{ijkl}u_{k,l}+C_{ijkl}\dot{u}_{k,l}-\beta_{ij}\theta)_{,j}, \tag{7.1}$$

$$c\dot{\theta}=(k^*_{ij}\alpha_{,i})_{,j}+(k_{ij}\theta_{,i})_{,j}-\beta_{ij}\dot{u}_{i,j}-\tau\beta_{ij}\ddot{u}_{i,j} \tag{7.2}$$

where

$$\alpha(\mathbf{x},t)=\alpha(\mathbf{x},0)+\int_0^t\theta(\mathbf{x},s)ds$$

is the thermal displacement.

We want to state a uniqueness and a instability results for the solutions of the problem determined by this system with boundary conditions

$$u_i(\mathbf{x},t)=0,\quad \theta(\mathbf{x},t)=0 \text{ and } \alpha(\mathbf{x},t)=0,\quad \mathbf{x}\in\partial B,\ \ t>0, \tag{7.3}$$

and initial conditions

$$u_i(\mathbf{x},0)=u_i^0(\mathbf{x}),\ \ \dot{u}_i(\mathbf{x},0)=v_i^0(\mathbf{x})\text{and } \ddot{u}_i(\mathbf{x},0)=w_i^0(\mathbf{x}),\quad \mathbf{x}\in B. \tag{7.4}$$

and

$$\theta(\mathbf{x},0)=\theta^0(\mathbf{x}) \text{ and } \alpha(\mathbf{x},0)=\alpha^0(\mathbf{x}),\quad \mathbf{x}\in B. \tag{7.5}$$

In this section we will be assuming k_{ij}, k^*_{ij} to be symmetric and positive (hypothesis (1.2), (3.6), (4.1)) and the same for $\overline{C}_{ijkl}$ (the symmetry can be deduced from (6.6) and its positivity is hypothesis (6.7)). Also, we assume the positivity of the mass density ρ (see (3.7)). We will be considering the same initial and boundary conditions as in the previous sections, together with null Dirichlet boundary conditions for α. Then, the analysis done in previous sections can be adapted by considering the energy equation

$$E_4(t)=E_4(0) \tag{7.6}$$

where

$$E_4(t)=\int_B\big(\rho(\dot{u}_i+\tau\ddot{u}_i)(\dot{u}_i+\tau\ddot{u}_i)+C^*_{ijkl}(u_{i,j}+\tau\dot{u}_{i,j})(u_{k,l}+\tau\dot{u}_{k,l})+\tau\overline{C}_{ijkl}\dot{u}_{i,j}\dot{u}_{k,l}+c\theta^2+k^*_{ij}\alpha_{,i}\alpha_{,j}\big)\,dv$$

$$+2\int_0^t\int_B\big(\overline{C}_{ijkl}\dot{u}_{i,j}\dot{u}_{k,l}+k_{ij}\theta_{,i}\theta_{,j}\big)\,dvds.$$

We proceed as in Section 6 and integrate (7.2) with respect to time, obtaining:

$$c\theta=(k^*_{ij}\varphi_{,i})_{,j}+(k_{ij}\alpha_{,i})_{,j}-\beta_{ij}u_{i,j}-\tau\beta_{ij}\dot{u}_{i,j} \tag{7.7}$$

where

$$\varphi(\mathbf{x},t)=\int_0^t\alpha(\mathbf{x},s)ds+\Upsilon(\mathbf{x})$$

and $\Upsilon(\mathbf{x})$ being a solution of

$$\left(k^*_{ij}\Upsilon_{,i}\right)_{,j}=\beta_{ij}(u^0_{i,j}+\tau\dot{u}^0_{i,j})+c\theta^0$$

with homogeneous Dirichlet boundary conditions.

We then define
(7.8)

$$F(t,\omega,t_0)=\int_B(\rho(u_i+\tau\dot u_i)(u_i+\tau\dot u_i)+k^*_{ij}\varphi_{,i}\varphi_{,j})dv+\int_0^t\int_B\left(\overline{C}_{ijkl}u_{i,j}u_{k,l}+k_{ij}\alpha_{,i}\alpha_{,j}\right)dvds+\omega(t+t_0)^2.$$

Observe that under the assumptions assumed in this section F is a strictly positive function.

Deriving F with respect to time we obtain

(7.9)

$$\begin{aligned}\frac{dF(t,\omega,t_0)}{dt}&=2\int_B(\rho(u_i+\tau\dot u_i)(\dot u_i+\tau\ddot u_i)+k^*_{ij}\varphi_{,i}\alpha_{,j})dv+2\int_0^t\int_B\left(\overline{C}_{ijkl}u_{i,j}\dot u_{k,l}+k_{ij}\alpha_{,i}\theta_{,j}\right)dvds\\&+\int_B\left(\overline{C}_{ijkl}u^0_{i,j}u^0_{k,l}+k_{ij}\alpha^0_{,i}\alpha^0_{,j}\right)dv+2\omega(t+t_0)\end{aligned}$$

and

$$\begin{aligned}\frac{d^2F(t,\omega,t_0)}{dt^2}&=2\int_B(\rho(\dot u_i+\tau\ddot u_i)^2+(u_i+\tau\dot u_i)(\ddot u_i+\tau\dddot u_i)+k^*_{ij}\alpha_{,i}\alpha_{,j}+k^*_{ij}\varphi_{,i}\theta_{,j})dv\\&+2\int_B\left(\overline{C}_{ijkl}u_{i,j}\dot u_{k,l}+k_{ij}\alpha_{,i}\theta_{,j}\right)dv+2\omega.\end{aligned}$$

We now multiply (7.1) by $u_i+\tau\dot u_i$ and (7.7) by θ, integrate each one on B, and use the resulting equalities into the previous expression, and we obtain:

$$\begin{aligned}\frac{d^2F(t,\omega,t_0)}{dt^2}&=2\int_B(\rho(\dot u_i+\tau\ddot u_i)^2-C^*_{ijkl}u_{k,l}(u_{i,j}+\tau\dot u_{i,j})-C_{ijkl}\dot u_{k,l}(u_{i,j}+\tau\dot u_{i,j})+k^*_{ij}\alpha_{,i}\alpha_{,j})dv\\&-2\int_B c\theta^2dv+2\int_B\overline{C}_{ijkl}u_{i,j}\dot u_{k,l}dv+2\omega.\end{aligned}$$

Substituting the energy equation (7.6) into the previous equality, we obtain the following alternative expression for $\frac{d^2F}{dt^2}$:

$$\begin{aligned}\frac{d^2F(t,\omega,t_0)}{dt^2}&=4\int_B(\rho(\dot u_i+\tau\ddot u_i)^2+k^*_{ij}\alpha_{,i}\alpha_{,j})dv\\&+4\int_0^t\int_B\left(\overline{C}_{ijkl}\dot u_{i,j}\dot u_{k,l}+k_{ij}\theta_{,i}\theta_{,j}\right)dvds+2(\omega-E_4(0)).\end{aligned}\tag{7.10}$$

Using expressions (7.8), (7.9) and (7.10), we can see that inequality (2.9) is satisfied with

$$\nu=2\int_B\left(\overline{C}_{ijkl}u^0_{i,j}u^0_{k,l}+k_{ij}\alpha^0_{,i}\alpha^0_{,j}\right)dv$$

Uniqueness of solutions can be obtained as well as their instability whenever we assume that $E_4(0)<0$ using the same arguments as in the previous sections.

Theorem 7.1. *Assume that the symmetry conditions* (1.2) *and* (6.6)*, and the positivity assumptions* (3.6)*,* (3.7)*,* (4.1) *and* (6.7) *hold. Consider the first initial-boundary-value problem* (7.1)*-*(7.2) *with the corresponding boundary and initial conditions* (7.3)*-*(7.5)*. Then,*

(i) *the first initial-boundary-value problem* (7.1)*-*(7.5) *has at most one solution.*
(ii) *If $E_4(0)<0$, the solution of* (7.1)*-*(7.5) *becomes unbounded in an exponential way.*

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Conflicts of interest

The authors declare that they have no conflict of interest.

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