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An Age-Structured Population Approach for the Mathematical Modeling of Urban Burglaries*

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Abstract. We propose a nonlinear model for the dynamics of urban burglaries which takes into account the deterring effect of the police. The model focuses on the timing of criminal activity rather than on the spatial spreading of burglaries and it is inspired in the age-dependent population dynamics. The structuring variables are the time elapsed between two consecutive offenses committed by a burglar or suffered by a house. The main ingredients of the model are the propensity of burglars to commit a crime and the rate at which houses are being burgled. These rates are taken as general as possible to allow different scenarios, including the widely used repeat victimization pattern. The dissuasive effect of the active police deployment is introduced by means of a memory term that depends on the number of the last committed burglaries. The asymptotic behavior of the model and the existence of a globally stable equilibrium are determined thanks to a suitable change of variables that involves a continuous rescaling of the time variable. This new approach provides some interesting analytic results on the equilibrium and the expected times between consecutive offenses. Numerical simulations are shown to illustrate these results.

Key words. structured population dynamics, history-dependent rates, expected time between criminal activities

AMS subject classifications. 92D25, 35Q92, 35L50, 65M25

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1. Introduction and statement of the problem.

Urban burglary is one of the most common offenses which police departments must deal with daily. Furthermore, this type of crime usually produces a strong sense of lack of public safety since the victims stop feeling safe in their own homes. However, despite police efforts and in part due to the global economic recession, the number of burglaries in Catalonia has increased (see [1]). In a recent *European Study Group with Industry* (see [31]) the Catalan Police Department set out the problem of finding suitable mathematical models to predict the evolution of urban burglaries in Catalonia. This paper is, in part, motivated by the research started during this workshop.

Police investigators often report that there seem to be specific periods of time where waves of burglaries take place. Furthermore, in large suburban areas where houses tend

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to be rather similar in their structure and, furthermore, are regularly aligned, hotspots of crime evolving in space and time have long been observed (see, for instance, [6, 8]). Some years ago these observations motivated the search for mathematical models to describe the evolution of criminal activity. One of the most celebrated models for residential burglaries was first proposed by Short et al. [39]. Their main hypothesis, based on studies like the one by Johnson, Bowers, and Hirschfield [27], is that residential burglars prefer to return to a previously burgled house, or a house nearby, in a short period of time after their last strike, in part because they have learned escape routes and also because they know what sort of valuables may be stolen. This hypothesis is usually denoted in the literature as *repeat* or, correspondingly, *near-repeat* victimization. Under this hypothesis, Short et al. [39] devised a model that relates the attractiveness of a house to burglaries with the number of criminals present at the house position (which can be thought of as a measure of the probability of the house being burgled). With this sort of victimization, the attractiveness of the house is elevated after a burglary and so is that of nearby houses. They start with a discrete model that assumes that the houses are aligned in a square lattice and they then derive a continuous model in terms of spatially reaction-diffusion nonlinear partial differential equations (PDEs) by letting the side of the squares of the lattice tend to zero. This agent-based model already captures the essential dynamics of hotspot formation.

In the last few years, there has been increasing interest in the mathematical modelling of crime and, more concretely, of burglaries (see, for instance, [7] or [15]). In particular, this model has since been extended and recast by many authors. For instance, Pitcher and Johnson in [37] include a random variable to account for the number of burglaries that occur at a given site, and Chaturapruek et al. [12] use Lévy flights to drive the criminals' motion which results on a nonlocal PDE system. In any case, the final set of differential equations results in being far from trivial and has indeed given rise to a number of publications devoted to the analysis and computation of the solutions of these systems.

These first models, however, do not consider any type of police intervention. Short et al. [40, 41] add a deterrence term to the equations to model the effect of police deployment. The deterrence function in this model is a function of space, and its effect is focused at the center of the hotspots. Pitcher in [36] considers a dynamic deterrence which represents the amount by which the attractiveness is reduced in the presence of police patrols. In this case the deterrence term depends on space and time and it is updated by means of a set of laws for the deployment of police written in terms of a PDE. Also Zipkin, Short, and Bertozzi [47] include deterrence in the Short model, as well as a function to account for the police resources. In this case, the rule of movement for police agents is such that the total number of offenses at a given time is minimized. They then write the model in terms of a set of PDEs coupled to a optimization problem. Some recent results on this approach can be found in [30, 10], both published in the last special number of EJAM on mathematical modeling of crime and security [EJAM, vol. 27, 2016].

In [33], Nuño, Herrero, and Primicerio do not use the agent-based model approach but suggest a system of three ordinary differential equations of predator-prey type to study the interaction between the owners, burglars, and police. Also, Beresticky and Nadal [4] formulate a PDE model which is also not agent-based but it also considers repeat victimization and exhibits hotspots. They also include the police intervention which reduces the strength of hotspots. Birks, Townsley, and Ryzhik [5] present a new agent-based model where again criminal agents move on a rectangular grid but, in this case, there is a set of targets randomly distributed throughout the domain instead of having a target at each node of the lattice. This is, indeed, the first work where a certain level of spatial heterogeneity is allowed.

The models that have just been described are based on a repeat and near-repeat victimization hypothesis. The phenomenon has been widely observed in domestic burglaries in cities of Europe, the United States, and Australia (see, for instance, [3] or [28]), and its existence has been exploited in numerous policing experiments to reduce crime. However, it has been also observed in [43] that near repeats exist mainly in suburbs containing homogeneous housing, i.e., in areas where dwellings exhibit similar characteristics and security features. On the other hand, a recent study examining repeat and near-repeat victimization of domestic burglaries in Belo Horizonte, Brazil [16] has revealed levels of this type of victimization that were much lower than those found in studies from other countries. Indeed, urban dwellings in Brazil are quite different from dwellings in western countries. According to the authors, many more domestic properties in Brazil have situational prevention measures, such as perimeter fences and security guards, implemented as standard to improve domestic safety. So, the opportunities to commit burglary are considered to be lower due to these higher levels of in-built situational crime prevention that fortify homes from would-be offenders. Therefore, analytic burglary models which are flexible enough to include also alternative patterns of victimization seem to be desirable.

Our goal in this paper is to postulate models for urban burglaries derived from the minimum number of hypotheses. In fact, the models will be simple enough to allow for the derivation of some explicit expressions. This is achieved by adopting a new approach inspired in age-structured population models (see [25, 42, 24, 45]). The latter study the time evolution of a population in terms of the physiological age of its individuals, which is relevant for the dynamics because death and fecundity rates are assumed to be age dependent. Following this approach, the populations of burglars and houses will be described in terms of densities with respect to a label or "age." For each burglar, this label indicates the time elapsed since the last offense (burglary age) while, for each house, it corresponds to the time passed since the house was burgled (victimization age). In some sense, this label can be thought of as a first step towards a certain heterogeneity in the population of houses, as well as in the population of burglars, which is something that makes urban burglaries different from standard suburban areas.

What plays the role of death and fecundity rates (equal to each other in our models) are, for burglars, the propensity of a burglar to commit a crime (recurrence rate), and, for houses, the rate at which houses are being burgled (victimization rate). Recurrence rate is, in fact, related to the so-called individual offending frequency in criminology, which is defined as "the rate of crimes by offenders who commit crimes at some nonzero rate" [11]. On the other hand, the tendency to commit burglaries in small groups rather than alone, also known as co-offending, is one of the distinctive behaviors in burglary [14, 26, 29]. In the model, this is translated into the fact that the total number of burglaries per unit of time will always be less than or equal to the total number of acting burglars per unit of time, being that the co-offending factor is the quantity that determines the relation between both rates. This factor, in turn, is assumed to depend on the mean vulnerability of houses. The underlying idea is simple: the lower the mean vulnerability of houses is, the higher the degree of co-offending required for successfully breaking into them.

Therefore, the main novelty of our model is the fact that we consider general functions for the recurrence rate and for houses' vulnerability rather than using a priori specific behavioral hypotheses, such as those related to repeat and near-repeat victimization, which limit the customization possibilities of the model. Fitting real data of a city or area to derive the corresponding recurrence and victimization rate functions is in itself an interesting and far from trivial statistical problem which is beyond the purpose of this work.

The paper is organized as follows. In section 2 we introduce the basic model ingredients and the equations that govern the dynamics of the densities of houses and burglars with no extra police deployment. We obtain its asymptotic behavior. In section 3 we extend this model by introducing a deterrence factor which modulates the propensity of burglars to commit the next offense. This factor is assumed to be a function of the number of burglaries that occurred during the last T units of time, hence introducing a memory into the system dynamics. We then show that this model may be directly related to the basic one in section 2 by suitably performing some changes of variables. We thus find a way to express a system with delay in terms of a simpler one without memory. In particular, this allows us to prove that all nonnegative solutions of the model with police deterrence tend to a unique equilibrium, whose expression is also derived. Section 4 is devoted to providing some numerical examples of the dependence of the main outputs of the model on the length T of the observation period and also on some other parameters of the system. These numerical examples show that one can easily analyze the effect of the police response on the number of burglaries, which allows us to explore different possible scenarios. Finally, discussion and further work are presented in section 5.

2. Burglars-Houses model without active police deployment. Let us start by introducing some notation and the basic model assumptions. Throughout the paper, $N(\tau_1, t)$ will denote the (nonnegative) density of burglars at time t that have offended τ_1 units of time ago, and $H(\tau_2, t)$ will denote the (nonnegative) density of houses at time t that have been burgled τ_2 units of time ago. So, we can think of τ_1 as the burglary age of a burglar, and of τ_2 as the victimization age of a house. Notice that in this paper, the term age has nothing to do with the chronological age of the individuals.

Neglecting demographic turnover and assuming a closed population of burglars acting on a specific geographic area, we consider that the total number of burglars remains constant in time and the same is assumed for the total number of houses:

$$N^0 := \int_0^\infty N(\tau, t)\, d\tau, \quad H^0 := \int_0^\infty H(\tau, t)\, d\tau \;\; \forall\, t \geq 0.$$

To derive the model equations, we need to establish how houses are burgled and how offenders act. First, we will assume that the victimization age τ_2 of a house strongly relates to its features as a target of a burglary. This assumption reflects the fact that the implementation of (updated) prevention measures as well as the value of the goods inside may depend on the time elapsed since it was burgled. In other words, τ_2 determines if a house is more or less attractive to burglars.

The victimization rate $\alpha(\tau_2, t)$ at time t of a house of age τ_2 is defined as the rate of burglaries committed at time t by co-offending groups per house of age τ_2 plus the rate of burglaries committed at time t by single burglars per house of age τ_2. By consistency of the model, the rate of burglaries of houses of age τ_2 at time t, $\alpha(\tau_2, t)H(\tau_2, t)$, is equal to the rate of single burglars plus the rate of co-offending groups, both of them acting on these houses. So, $\int_0^\infty \alpha(\tau, t)H(\tau, t)d\tau$ is always less than or equal to the rate of active burglars, $A(t)$, with the equality being attained only when all burglars act alone. Let us denote by $\rho(\tau_2, t)$ the proportionality factor that relates $\alpha(\tau_2, t)$ to the per-house rate of active burglars, $A(t)/H^0$, that is, $\alpha(\tau_2, t) = \rho(\tau_2, t)\frac{A(t)}{H^0}$ with $\rho(\tau_2, t) \geq 0$. Note that, by construction, the rate of burglaries at time t is

$$\int_0^\infty \alpha(\tau, t)H(\tau, t)d\tau = \overline{\rho}(t)A(t)$$

with $\overline{\rho}(t) = \int_0^\infty \rho(\tau, t)H(\tau, t)/H^0 d\tau \leq 1$ being the co-offending factor since it relates the rate of active burglars to the (less than or equal to) rate of burglaries. To derive a more precise expression for $\alpha(\tau_2, t)$, we need to describe how burglars act.

The vulnerability function, $\alpha_0(\tau_2)$, reflects the ease of breaking into a house of age τ_2. Somehow, it incorporates the information about security features of a house that we have assumed to be strongly related to its victimization age. So the mean vulnerability of houses when their density of $H(\tau_2, t)$ is

$$\overline{\alpha}_0(t) := \int_0^\infty \alpha_0(\tau)\frac{H(\tau, t)}{H^0}\, d\tau, \tag{1}$$

where $H(\tau_2, t)/H^0$ is the probability density of finding a house of age τ_2 at time t. We assume that $\alpha_0(\tau_2)$ is a bounded continuous function with $\alpha_0(\tau_2) > 0$ for $\tau_2 > 0$ and, without loss of generality, that $\alpha_0(\tau_2) \leq 1$ (the maximum vulnerability of a house is set to 1). The mean vulnerability will determine how burglars see, on average, the set of houses and, hence, their criminal behavior.

The propensity of a burglar to commit a crime is introduced by means of the function $f(\tau_1, \overline{\alpha}_0)$, the per capita recurrence rate of burglars of burglary age τ_1 when the mean vulnerability of houses is $\overline{\alpha}_0$. It measures their tendency to break into a house under an ordinary police deployment, i.e., without an active police response to changes in criminal activity. The rationale behind the depedence of f on τ_1 and $\overline{\alpha}_0$ is given by the classic notion in criminology that for a burglary to occur, a motivated burglar must find a suitable house [13]. Here, the propensity level is assumed to be a function of the burglary age τ_1, whereas the accessibility to target sites is represented by the mean vulnerability of houses, $\overline{\alpha}_0$. In what follows we shall assume $f(\tau_1, \overline{\alpha}_0)$ to be a nonnegative bounded continuous function with $\liminf_{\tau_1\to\infty} f(\tau_1, \overline{\alpha}_0) > 0$ $\forall\, \overline{\alpha}_0 > 0$ and, moreover, to be continuously differentiable with respect to $\overline{\alpha}_0$. The strict positivity of the previous limit implies that any burglar, sooner or later, will again commit an offense as long as there are vulnerable targets. Finally, we shall also assume that $f(\tau_1, 0) = 0$ $\forall \tau_1 \geq 0$, which says that it is not possible for burglars to commit a successful offense when there are no vulnerable houses. Therefore, at time t, the rate of active burglars is then given by

$$A(t) = \int_0^\infty f(\tau, \overline{\alpha}_0(t))N(\tau, t)\, d\tau.$$

To illustrate how information about co-offending can be included in $\alpha(\tau_2, t)$ through ρ, we assume that, whenever one burglar decides to commit a crime, he/she may choose to join some others depending on their burglary ages and, also, on the mean vulnerability of the houses at each time. Actually, it has been reported that the majority of burglars decide details such as the precise target to hit and methods to be used after they have formed the intention to burgle (see [23] and references therein). Then, after the crime, co-offending groups are assumed to break up. This is why, from now on, we will work with the density of active burglars per unit of time, $f(\tau, \overline{\alpha}_0)N(\tau, t)$.

Although these are simple rules, a general analytic description of co-offending as a process of group formation leading to a distribution of groups of different sizes is certainly beyond the scope of the present paper because of its own complexity (see, for instance, [20]). Instead, we restrict our attention to the simpler process of formation of co-offending pairs. In one-sex age-structured populations, pair formation is described as a dynamic process involving affinities among different types of individuals [22]. In the current context, the affinity m between two active burglars will be assumed to be a nonnegative bounded continuous function of their burglary ages τ_1^a and τ_1^b, and of $\overline{\alpha}_0(t)$. Moreover, since co-offending pairs split up after committing a crime, we will approximate the density of single active burglars required to compute the pair formation rate by the density of active burglars, i.e., without taking into account those who already form a pair of co-offending burglars. Under such an approximation, the rate (of creation) of co-offending pairs of burglars breaking into a house is

$$\mathcal{P}(t) = \int_0^\infty \int_0^\infty \frac{m(\tau_1^a, \tau_1^b; \overline{\alpha}_0(t))\, f(\tau_1^a, \overline{\alpha}_0(t))N(\tau_1^a, t)\, f(\tau_1^b, \overline{\alpha}_0(t))N(\tau_1^b, t)}{2\int_0^\infty f(\tau, \overline{\alpha}_0(t))N(\tau, t)\, d\tau}\, d\tau_1^b\, d\tau_1^a.$$

Note that this rate is a homogeneous function of degree 1 with respect to the density of active burglars per unit of time, $f(\tau, \overline{\alpha}_0)N(\tau, t)$, which is one of the properties that are usually required for pair formation laws [22]. Moreover, since there cannot be more co-offending pairs per unit of time than half of the total number of active burglars, a consistency relationship that is required for the total number of co-offending pairs created per unit of time is $\mathcal{P}(t) \leq \frac{1}{2}\int_0^\infty fN d\tau$, which is the one-sex version of condition (4) in [46] for a general two-sex age-structured population model. To fulfill this consistency requirement we will assume $m(\tau_1^a, \tau_1^b; \overline{\alpha}_0) \leq 1\ \forall\, \tau_1^a, \tau_1^b, \overline{\alpha}_0 \geq 0$.

Now, recalling the definition of the victimization rate $\alpha(\tau_2, t)$ and taking into account the per-house rate of burglaries committed by co-offending pairs, $\mathcal{P}(t)/H^0$, and the per-house rate of burglaries committed by burglars acting alone, $(A(t) - 2\mathcal{P}(t))/H^0$, we have that the burglary rate at time t of houses of age τ_2 is given by

$$\begin{aligned}\alpha(\tau_2, t)H(\tau_2, t) &= \eta(\alpha_0(\tau_2), t)\left(\frac{\mathcal{P}(t)}{H^0} + \frac{\left(\int_0^\infty fN\, d\tau - 2\mathcal{P}(t)\right)}{H^0}\right)H(\tau_2, t)\\ &= \eta(\alpha_0(\tau_2), t)\left(\int_0^\infty fN\, d\tau - \mathcal{P}(t)\right)\frac{H(\tau_2, t)}{H^0},\end{aligned}$$

that is, the product of the sum of the rate of co-offending burglars and the corresponding fraction of single active burglars per unit of time, times an allocation function $\eta(\alpha_0(\tau_2), t) \geq 0$ of these burglars among houses which is assumed to depend on the vulnerability of these houses. To distribute active burglars among all houses, this allocation function must satisfy that $\int_0^\infty \eta(\alpha_0(\tau),t)H(\tau,t)/H^0 d\tau = 1\ \forall t$. In particular, a uniform distribution of active burglars (acting alone and in pairs) among houses, i.e., a distribution independent of the victimization age of houses, follows if $\eta(\alpha_0(\tau_2),t) \equiv 1$. An example of such an allocation function is given by $\eta(\alpha_0(\tau_2),t) = (\alpha_0(\tau_2)+a)/(\overline{\alpha}_0(t)+a)$, $a \geq 0$ which, in the limit case of $a \to \infty$, satisfies $\eta(\alpha_0(\tau_2),t) \equiv 1$.

If we simplify the creation of co-offending pairs by assuming that the affinity among offenders only depends on the mean vulnerability of the houses, i.e., $m = m(\overline{\alpha}_0)$, then $\mathcal{P}(t) = m(\overline{\alpha}_0(t))\frac{1}{2}\int_0^\infty fN$, and, hence,

$$\alpha(\tau_2,t) = \eta(\alpha_0(\tau_2),t)\left(1 - \frac{m(\overline{\alpha}_0(t))}{2}\right)\frac{1}{H^0}\int_0^\infty fN\,d\tau. \tag{2}$$

So, we have a relationship between the proportionality factor ρ at time t and the affinity m between burglars to co-offend in pairs, namely,

$$\rho(\tau_2,t) = \eta(\alpha_0(\tau_2),t)\left(1 - \frac{m(\overline{\alpha}_0(t))}{2}\right).$$

In this case, according to the hypotheses on m and η, the co-offending factor $\overline{\rho}$ satisfies

$$\frac{1}{2} \leq \overline{\rho}(\overline{\alpha}_0(t)) = 1 - \frac{m(\overline{\alpha}_0(t))}{2} \leq 1.$$

The lower bound $1/2$ corresponds to the case where all active burglars co-offend in pairs $(m = 1)$ whereas, when $\overline{\rho}(\overline{\alpha}_0(t)) = 1$, all of them act alone $(m = 0)$.

To deal with co-offending groups having more than two members and victimization rates depending on the age of houses, from now on we will assume that $\rho(\tau_2,\overline{\alpha}_0)$ is a continuous function of τ_2 and continuously differentiable with respect to $\overline{\alpha}_0$, with $\rho(\tau_2,\overline{\alpha}_0) \geq 0$ and $\overline{\rho}(\overline{\alpha}_0) \leq 1$. Note that larger co-offending groups amount to lower bounds for $\overline{\rho}(\overline{\alpha}_0)$ smaller than $1/2$. Moreover, it satisfies $\liminf_{\tau_2\to\infty} \rho(\tau_2,\overline{\alpha}_0) > 0\ \forall\,\bar{\alpha}_0 > 0$, which means that, sooner or later, all houses of the system are going to be burgled, and also it is increasing in $\overline{\alpha}_0$. Finally, introducing the expression of $A(t)$ into the victimization rate α, the latter becomes

$$\alpha(\tau_2, N(\cdot,t), \overline{\alpha}_0(t)) = \rho(\tau_2,\overline{\alpha}_0(t))\frac{1}{H^0}\int_0^\infty f(\tau,\overline{\alpha}_0(t))N(\tau,t)d\tau. \tag{3}$$

2.1. The model equations. These modeling assumptions imply that the dynamics for the densities $N(\tau_1,t)$ and $H(\tau_2,t)$ are described in terms of a predator-prey type of interaction between houses and burglars, where the former are the prey and the latter the predators. When a burglar strikes or when a house is burgled, their age resets to $\tau_1 = 0$ and $\tau_2 = 0$, respectively, and the burglary model is given by the following nonlinear system of first-order PDEs with nonlocal boundary conditions (where both ages are denoted using different subscripts for clarity of presentation):

$$
\left\{
\begin{array}{rcl}
\partial_t N(\tau_1,t)+\partial_{\tau_1}N(\tau_1,t) & = & -f(\tau_1,\overline{\alpha}_0(t))N(\tau_1,t),\\
\partial_t H(\tau_2,t)+\partial_{\tau_2}H(\tau_2,t) & = & -\alpha(\tau_2,N(\cdot,t),\overline{\alpha}_0(t))H(\tau_2,t),\\
N(0,t) & = & \displaystyle\int_0^\infty f(\tau,\overline{\alpha}_0(t))N(\tau,t)\,d\tau,\\
H(0,t) & = & \displaystyle\int_0^\infty \alpha(\tau,N(\cdot,t),\overline{\alpha}_0(t))H(\tau,t)\,d\tau,
\end{array}
\right. \tag{4}
$$

endowed with the initial conditions $N(\tau_1,0)=N_0(\tau_1)$ and $H(\tau_2,0)=H_0(\tau_2)$, being both nonnegative functions in $L^1(0,\infty)$ such that

$$\int_0^\infty N_0(\tau)\,d\tau = N^0 \quad\text{and}\quad \int_0^\infty H_0(\tau)\,d\tau = H^0.$$

As expected from the reset of the ages of burglars and houses, integrating both PDEs with respect to τ, using the boundary conditions and the fact that, for each t, $N(\tau,t)$ and $H(\tau,t)$ are integrable functions, it follows that the solution of this initial value problem satisfies $\int_0^\infty N(\tau,t)\,d\tau = N^0$ and $\int_0^\infty H(\tau,t)\,d\tau = H^0\ \forall\, t\geq 0$.

Observe that from (3) the boundary condition on H can be written as

$$H(0,t)=\overline{\rho}(t,\overline{\alpha}_0(t))N(0,t)$$

with the co-offending factor $\overline{\rho}(t,\overline{\alpha}_0(t)) = \int_0^\infty \rho(\tau,\overline{\alpha}_0(t))H(\tau,t)/H^0 d\tau$. If $\overline{\rho}=\overline{\rho}(\overline{\alpha}_0(t))$ with $\overline{\rho}(1)=1$, and all the houses are equally vulnerable and their vulnerability is maximum ($\alpha_0\equiv 1$), then $H(0,t)=N(0,t)$, i.e., the number of burglaries per unit of time is equal to the rate of active burglars.

Under the hypotheses on ρ, f, and α_0, the well-posedness of (4) and the existence of a global solution of this system follow from the proof given in [44] for a general model of nonlinearly interacting age-dependent populations, once the integrals of the solution in that proof are replaced with the weighted integrals of $N(\tau_1,t)$ and $H(\tau_2,t)$ appearing in (1) and (3), and it is used that, under the hypotheses on $f(\tau_1,\overline{\alpha}_0)$, the weighted integral of $N(\tau_1,t)$ is a Lipschitzian function of the weighted integral of $H(\tau_2,t)$.

2.2. Asymptotic behavior of solutions. From now on in this section, we will assume that $\alpha(\tau_2,N(\cdot,t),\overline{\alpha}_0)$ is given by (3) with

$$\rho(\tau_2,\overline{\alpha}_0)=\rho_0(\tau_2)\rho_1(\overline{\alpha}_0). \tag{5}$$

Similarly, we will also assume that the effect of the mean vulnerability on the recurrence rate of an offender is independent of its burglary age, i.e.,

$$f(\tau_1,\overline{\alpha}_0)=f_0(\tau_1)f_1(\overline{\alpha}_0), \tag{6}$$

where $f_0(\tau_1)$ is the inherent recurrence rate of a burglar of burglary age τ_1 and represents its natural propensity for crime, and $f_1(\overline{\alpha}_0)$ is the effect of the mean vulnerability of houses on burglars' activity.

The hypotheses on ρ and f imply that $\rho_0(\tau_2)$ and $f_0(\tau_1)$ are nonnegative continuous bounded functions with $\liminf_{\tau_2\to\infty}\rho_0(\tau_2) > 0$ and $\liminf_{\tau_1\to\infty} f_0(\tau_1) > 0$ and, moreover, that $\rho_1(\overline{\alpha}_0)$ and $f_1(\overline{\alpha}_0)$ are continuously differentiable with $f_1(0) = 0$. Here, we will also assume that $\rho_1(\overline{\alpha}_0)$ and $f_1(\overline{\alpha}_0)$ are strictly positive for $\overline{\alpha}_0 > 0$, and that the vulnerability function $\alpha_0(\tau) \geq \alpha_0^0 > 0\ \forall\, \tau \geq 0$. In particular, this implies $\overline{\alpha}_0(t) \geq \alpha_0^0\ \forall\, t$, and, hence, that $\rho_{1,0} := \min_{\alpha_0^0\leq x\leq 1}\{\rho_1(x)\} > 0$ and $f_{1,0} := \min_{\alpha_0^0\leq x\leq 1}\{f_1(x)\} > 0$. From a modeling point of view, one thinks of $\rho_1(\overline{\alpha}_0)$ and $f_1(\overline{\alpha}_0)$ as increasing functions, tending to 1 as $\overline{\alpha}_0 \to 1$. In this case, $\rho_{1,0} = \rho_1(\alpha_0^0) > 0$ and $f_{1,0} = f_1(\alpha_0^0) > 0$.

Now, let $(N(\tau_1, t), H(\tau_2, t))$ be a solution of (4), let $\overline{\alpha}_0(t)$ be the mean vulnerability computed from this solution according to (1) for $t \geq 0$, and let us define $\overline{\alpha}_0(t) = \overline{\alpha}_0(0)$ for $t < 0$. Moreover, let us consider the new time variable

$$t' = \int_0^t f_1(\overline{\alpha}_0(s))\, ds,$$

and the new densities $(\bar{N}, \bar{H})$ of burglars and houses defined by

$$N(\tau, t) = f_1(\overline{\alpha}_0(t-\tau))\bar{N}(\tau, t)\ , \quad H(\tau, t) = f_1(\overline{\alpha}_0(t-\tau))\bar{H}(\tau, t)\ .$$

Taking into account that the age τ of a burglar (and similarly for a house) is the difference of two times, namely, the current time t minus the time t_0 of his/her last offense (or the time the house was burgled), i.e., $\tau = t - t_0$, the transformed age τ' is defined as the difference of the corresponding transformed times: $\tau'(\tau, t) := t' - t_0' = \int_{t-\tau}^{t} f_1(\overline{\alpha}_0(s))\, ds$. Note that the hypotheses on $f_1(\overline{\alpha}_0)$ and $\alpha_0(\tau)$ guarantee that $dt'/dt = f_1(\overline{\alpha}_0(t)) \geq f_{1,0} > 0$ which, in turn, implies that the new time t' never collapses and, so, $t' \to \infty$ as $t \to \infty$.

Introducing these new variables into (4), this system is transformed into an asymptotically autonomous one such that the dynamics of $\bar{N}(\tau', t')$ are decoupled from $\bar{H}(\tau', t')$, with the total number of burglars and houses still being equal to N^0 and H^0, respectively. Such a decoupling allows us to prove, by means of a second time variable $t'' = \int_0^{t'} \rho_1(\overline{\alpha}_0(s))ds$, that this new system has a unique equilibrium $(\bar{N}^*(\tau'), \bar{H}^*(\tau'))$ which attracts all of the trajectories. This convergence and the fact that dt'/dt and dt''/dt' have positive lower bounds $f_{1,0}$ and $\rho_{1,0}$, respectively, imply that the solutions of (4) also tend to a unique equilibrium $(N^*(\tau), H^*(\tau))$ (see the appendix for details).

2.3. The equilibrium.

After the convergence of a unique stationary solution $(N^*(\tau), H^*(\tau))$ is guaranteed when f and ρ factorize, it follows that this equilibrium is given by

$$N^*(\tau_1) = N^0 \frac{\Pi_b(\tau_1)}{\int_0^\infty \Pi_b(\tau)\, d\tau}, \qquad H^*(\tau_2) = H^0 \frac{\Pi_h(\tau_2)}{\int_0^\infty \Pi_h(\tau)\, d\tau}, \tag{7}$$

where

$$\Pi_b(\tau_1) := \exp\left(-\int_0^{\tau_1} f(\tau, \overline{\alpha}_0^*)\, d\tau\right) = \exp\left(-f_1(\overline{\alpha}_0^*)\int_0^{\tau_1} f_0(\tau)\, d\tau\right)$$

is the probability that a burglar remains inactive up to time τ_1, and

$$\mathit{\Pi}_h(\tau_2) := \exp\left(-\int_0^{\tau_2} \alpha(\tau, N^*(\cdot), \overline{\alpha}_0^*)\, d\tau\right)$$

is the probability of a house not being burgled up to time τ_2. In particular, from the expression of α, the assumption for ρ given by (5), and the boundary condition for $N^*(\tau_1)$, it follows that

$$\mathit{\Pi}_h(\tau_2) = \exp\left(-\frac{N^0}{H^0}\frac{\rho_1(\overline{\alpha}_0^*)\int_0^{\tau_2}\rho_0(\tau)\,d\tau}{\int_0^\infty \mathit{\Pi}_b(\tau)\,d\tau}\right),$$

where $\overline{\alpha}_0^*$ is the mean vulnerability of houses at equilibrium, i.e., the one resulting after replacing $H(\tau_2, t)$ with $H^*(\tau_2)$ in (1). Therefore, (7) is not an explicit expression of the equilibrium because both $\mathit{\Pi}_b(\tau_1)$ and $\mathit{\Pi}_h(\tau_2)$ depend on $\overline{\alpha}_0^*$.

To compute the equilibrium we consider the equation in $\overline{\alpha}_0^*$ given by (1) after plugging $H^*(\tau_2)$, namely,

$$\overline{\alpha}_0^* = \frac{\displaystyle\int_0^\infty \alpha_0(\tau)\mathit{\Pi}_h(\tau, \overline{\alpha}_0^*)\,d\tau}{\displaystyle\int_0^\infty \mathit{\Pi}_h(\tau, \overline{\alpha}_0^*)d\tau},$$

where the dependence of $\mathit{\Pi}_h$ on $\overline{\alpha}_0^*$ is explicit for the clarity of presentation. Note that the convergence of the solutions of (4) to a globally stable equilibrium implies that this equation for $\overline{\alpha}_0^*$ must have a unique solution in $(0, 1)$.

Under the hypotheses on $f(\tau_1, \overline{\alpha}_0)$, the *expected time between two consecutive offenses committed by the same burglar* is given by

$$\int_0^\infty \tau f(\tau, \overline{\alpha}_0^*)\mathit{\Pi}_b(\tau)\,d\tau = \int_0^\infty \mathit{\Pi}_b(\tau)\,d\tau < \infty,$$

which, as expected, decreases when the propensity of a burglar to break into a house increases.

Similarly, under the definition of $\alpha(\tau_2, N(\cdot, t), \overline{\alpha}_0)$ and the hypotheses on $\rho(\tau_2, \overline{\alpha}_0)$, it follows that

$$\int_0^\infty \tau\alpha(\tau, N^*(\cdot), \overline{\alpha}_0^*)\mathit{\Pi}_h(\tau)\,d\tau = \int_0^\infty \mathit{\Pi}_h(\tau)\,d\tau < \infty, \tag{8}$$

which is the *expected time between two consecutive burglaries of the same house.* Note that this mean time decreases when the total number of burglars N^0 increases, and increases with the total number of houses H^0 and with the expected time between burglaries committed by the same burglar, $\int_0^\infty \mathit{\Pi}_b(\tau)\,d\tau$, as one would expect.

Interestingly, when $\rho(\tau, \overline{\alpha}_0^*) = \rho(\overline{\alpha}_0^*)$ as in the example of co-offending in pairs with uniform distribution of burglars ($\eta(\tau_2, t) \equiv 1$), it immediately follows that, at equilibrium, the expected time between consecutive burglaries of a house is proportional to the expected time between two consecutive offenses committed by the same burglar, with the ratio of the former to the latter being equal to

$$\frac{1}{\rho(\overline{\alpha}_0^*)}\frac{H^0}{N^0}. \tag{9}$$

Finally, from (7) evaluated at $\tau = 0$ it follows that, at equilibrium, the number of active burglars per unit of time is equal to the total number of burglars divided by the expected time between two consecutive offenses, and that the rate of burglaries equals the total number of houses divided by the expected time between two consecutive burglaries of a given house; that is,

$$N^*(0) = \frac{N^0}{\int_0^\infty \Pi_b(\tau)\,d\tau}, \qquad H^*(0) = \frac{H^0}{\int_0^\infty \Pi_h(\tau)\,d\tau}. \tag{10}$$

3. Burglars-Houses model with active police deployment. In this second model, we consider a modification of system (4) in order to take into account an active police response to criminal activity. With this aim, we introduce an extra factor $0 < D_T \leq 1$ (depending on a certain observation period of time of length T) which modulates the recurrence rate of burglars producing a *deterred* recurrence rate given by $D_T \cdot f$. Since one wants the action of the police to be dissuasive, an increase in police resources must decrease the value of D_T. Moreover, because police reaction usually takes into account recently occurred events, we assume D_T to be history dependent. Concretely, the following form for D_T will be considered,

$$D_T\left(H\left(0,\cdot\right),t\right) := F\left(\int_{t-T}^{t} H(0,s)e^{-\xi(t-s)}\,ds\right),$$

where F is a strictly decreasing differentiable function of $\int_{t-T}^{t} H(0,s)e^{-\xi(t-s)}\,ds$, the weighted number of burglaries that occurred during the last T units of time, with $F(0) = 1$ and $F(x) \to 0$ as $x \to \infty$. The exponential weight is a discount factor with ξ being the discount rate. This term reflects the idea that recently committed burglaries have a higher impact on the current police response than those within the observation period that are more distant in the past. The decreasing behavior of F implies that if the number of burglaries within the observation period experiences a dramatic increase, then the deterred recurrence rate of burglars will be significantly reduced due to the dissuasive action of police. On the other hand, if the length of the observation period T is set to zero, the police does not take into account any past event and, hence, the recurrence rate does not change. This case corresponds to the situation described in the previous model where no extra police deployment was taken into account ($D_T \equiv 1$). Observe that, for simplicity, we assume that burglars of different burglary ages are equally deterred, since D_T is independent of τ_1.

We now deal with the per-house victimization rate, which in the previous model was denoted by $\alpha(\tau_2, N(\cdot,t), \overline{\alpha}_0(t))$ and was defined in (3). According to what we have explained before, we modify it by introducing the deterred recurrence rate, $D_T \cdot f$, which gives the *deterred victimization rate* of a house of age τ_2,

$$D_T(H(0,\cdot),t)\alpha(\tau_2, N(\cdot,t), \overline{\alpha}_0(t)).$$

This rate now depends not only on the density of burglars, $N(\tau_1, t)$, and on the current mean vulnerability of houses, $\overline{\alpha}_0(t)$, but also on the rate of new burglarized houses, $H(0,t)$, over a time period of length T through the functional $D_T\left(H(0,\cdot),t\right)$.

Neglecting demographic turnover and assuming a closed population of burglars, we again consider a constant total population of burglars, N^0, and houses, H^0. With the same type of interaction between houses and burglars as in the previous model, but now taking into account the above described dissuasive effect of the police, one obtains the following initial boundary value problem with delay:

$$
\left\{
\begin{array}{rcl}
\partial_t N(\tau_1,t)+\partial_{\tau_1}N(\tau_1,t) & = & -D_T(H(0,\cdot),t) f(\tau_1,\overline{\alpha}_0(t))N(\tau_1,t),\\
\partial_t H(\tau_2,t)+\partial_{\tau_2}H(\tau_2,t) & = & -D_T(H(0,\cdot),t)\alpha(\tau_2,N(\cdot,t),\overline{\alpha}_0(t))H(\tau_2,t),\\
N(0,t) & = & D_T(H(0,\cdot),t)\displaystyle\int_0^\infty f(\tau,\overline{\alpha}_0(t))N(\tau,t)\,d\tau,\\
H(0,t) & = & D_T(H(0,\cdot),t)\displaystyle\int_0^\infty \alpha(\tau,N(\cdot,t),\overline{\alpha}_0(t))H(\tau,t)\,d\tau,
\end{array}
\right.
\tag{11}
$$

endowed with the initial conditions $N(\tau_1,0)=N_0(\tau_1)$ and $H(\tau_2,0)=H_0(\tau_2)$, both nonnegative functions in $L^1(0,\infty)$, and initial history $H(0,t)=h_0(t)$ for $-T\le t<0$, a continuous bounded function in $[-T,0)$.

From the previous boundary conditions and definition (3) follows the same conclusion as in the model without active police deployment: $H(0,t)=\overline{\rho}(t,\overline{\alpha}_0(t))N(0,t)$ and if $\overline{\rho}=\overline{\rho}(\overline{\alpha}_0(t))$ with $\overline{\rho}(1)=1$, then $H(0,t)=N(0,t)$ when all the houses are equally vulnerable and their vulnerability is maximum ($\alpha_0(\tau_2)\equiv 1$), i.e., the number of burglaries per unit of time is equal to the rate of active burglars because then co-offending is not required.

Similar delayed PDEs are found in models of age-dependent population dynamics when there is a self-regulatory mechanism in the birth process which does not operate instantaneously, but responds to the recent history of the population over a given time interval (history-dependent birth rate) [17, 18, 19, 35]. One also finds noninstantaneous regulatory mechanisms in models for the dynamics of neural networks where the age of a neuron represents the time elapsed since its last discharge, and a delay is introduced in the computation of the global neural activity to take into account the electrical pulses created by neurons that have fired previously [34].

The study of the well-posedness of problem (11) falls outside the scope of this paper. However, it is easy to see that the proof goes along similar lines as the one for n-species model presented in [38]. In turn, the latter is a straightforward generalization of the proof in [18] for the one-species model with history-dependent rates which is based on the classical method of reducing the original problem to a renewal integral equation using the expression of the solution obtained after integrating the first-order PDE governing the population dynamics along characteristics [21].

3.1. The rescaled problem. The previous model with dynamic police deterrence can be transformed into a model with constant deterrence equivalent to (4), i.e., with $D_T\equiv 1$. This will allow us to obtain the asymptotic behavior of the solutions of the former. The idea is to introduce a continuous rescaling of the time variable in such a way that the decrease in the recurrence and victimization rates caused by D_T is translated into a transformed problem with a "slower" time variable where the total number of burglars and houses is preserved.

Given a solution $(N(\tau,t), H(\tau,t))$ of (11) and recalling that $H(0,t) = h_0(t)\ \forall\, t \in [-T, 0)$, let us introduce the new variables $(\bar{N}, \bar{H})$ for the population densities and a new time t' by

$$\begin{aligned} N(\tau,t) &= \varphi_T(t-\tau)\bar{N}(\tau,t),\\ H(\tau,t) &= \varphi_T(t-\tau)\bar{H}(\tau,t),\\ t' = \Psi_T(t) &= \int_0^t \varphi_T(s)\,ds, \end{aligned} \tag{12}$$

with

$$\frac{dt'}{dt} = \varphi_T(t) = \begin{cases} F\left(\int_{t-T}^t H(0,s)e^{-\xi(t-s)}ds\right), & t \geq 0,\\ F\left(\int_{-T}^t h_0(s)e^{-\xi(t-s)}ds\right), & -T \leq t < 0,\\ 1, & t < -T. \end{cases}$$

Note that the introduction of the densities $\bar{N}(\tau,t)$ and $\bar{H}(\tau,t)$ is made through the factor $\varphi_T(t-\tau)$ which requires the knowledge of the existing police deterrence D_T at $t-\tau$, the time of the last offense for a burglar of age τ, and of the last burglary for a house of age τ. Note that this time goes from $-\infty$ to t because the ages of burglars and houses go from 0 to ∞. Therefore, the rescaling of time must be defined for periods of time in the past that go beyond $-T$. However, there is no information about the number of burglaries before $t = -T$ because the initial history $h_0(t)$ is defined on $[-T, 0)$. This implies that D_T can only be computed from the number of burglaries occurring from $-T$ onwards and, hence, $\varphi_T(t)$ must be defined piecewise accordingly.

Moreover, recalling again that the age of a burglar (and similarly for a house) is just the difference of two times, namely, the current time t minus the time t_0 of his/her last offense (or the time a house was burgled), i.e., $\tau = t - t_0$, the transformed age τ' is defined to be the difference of the corresponding transformed times, namely,

$$\tau'(\tau,t) := t' - t'_0 = \Psi_T(t) - \Psi_T(t-\tau) \tag{13}$$

and, reciprocally, we have that $\tau(\tau',t') = \Psi_T^{-1}(t') - \Psi_T^{-1}(t'-\tau')$.

Introducing these new variables into system (11) and using that, from the definition of τ',

$$\frac{d\tau'}{d\tau} = \varphi_T(t-\tau),$$

it follows that the transformed population densities $\bar{N}'(\tau',t') := \bar{N}(\tau(\tau',t'), t(t'))$ and $\bar{H}'(\tau',t') := \bar{H}(\tau(\tau',t'), t(t'))$ satisfy the model without dynamic police deterrence ($D_T \equiv 1$) given by (4).

The main property of this change of variables is that history dependence determines how the time is rescaled. This is why D_T does not appear explicitly in the formulation of the new system. The new time $t' = \Psi_T(t)$ is, in fact, a contraction of the original one that takes into account the reduction of burglars' activity caused by an increased police deterrence. Precisely, according to the assumptions on F, f, and ρ, and using the expression of $H(0,t)$ given by the boundary condition in (11), the new time satisfies that

$$F\left(f^m N^0(1-e^{-\xi T})/\xi\right) t \le \Psi_T(t) \le t, \tag{14}$$

with f^m being an upper bound of f. The lower bound guarantees that the new time t' always increases with t, that is, it never collapses even when burglars' activity is very high.

History dependence also determines, by means of the initial history, the transformed initial condition $(\bar N'(\tau',0), \bar H'(\tau',0))$ obtained from $(N_0(\tau), H_0(\tau))$ using (12). More precisely, the information of the initial history $h_0(t)$, $t \in [-T,0)$, is embodied in $(\bar N'(\tau',0), \bar H'(\tau',0))$ for $\tau' \in [0,T')$ with $T' := -\Psi_T(-T) = \int_{-T}^{0} F\left(\int_{-T}^{\eta} h_0(s)e^{\xi s}\,ds\right) d\eta$. Note that this range of values of τ' corresponds to $\tau \in [0,T)$ in the original system. For $\tau \ge T$, since $\varphi_T(-\tau) = 1$ it follows from (13) that $\tau' = T' + (\tau - T) \ge T'$ and, hence, the relation between the initial conditions of the two models is simply given by $(\bar N'(\tau',0), \bar H'(\tau',0)) = (N_0(T+\tau'-T'), H_0(T+\tau'-T'))$.

Finally, the previous relationship between ages can be used to see that, as expected, the total numbers of burglars and houses are conserved after rescaling time, that is,

$$\int_0^\infty \bar N'(\tau',t')\,d\tau' = \int_0^\infty \bar N(\tau,t)\,\varphi_T(t-\tau)\,d\tau = \int_0^\infty N(\tau,t)\,d\tau = N^0$$

and

$$\int_0^\infty \bar H'(\tau',t')\,d\tau' = \int_0^\infty \bar H(\tau,t)\,\varphi_T(t-\tau)\,d\tau = \int_0^\infty H(\tau,t)\,d\tau = H^0.$$

3.2. Asymptotic behavior. From now on in this section, let us assume the same hypotheses on ρ and f as in subsection 2.2 (in particular, that ρ and f are given by (5) and (6), respectively). In what follows, we will prove that inequalities in (14), and the equivalence between the original system (11) and the transformed one satisfied by $(\bar N', \bar H')$, imply that the asymptotic behavior of the solutions to (11) is the same as the asymptotic behavior of the solutions to (4).

Precisely, recalling the definition of $\bar H'$, the second line in (12) can be written in terms of t' and τ' as

$$\frac{H(\tau(\tau',t'),t(t'))}{\varphi_T\left(t(t') - \tau(\tau',t')\right)} = \bar H'(\tau',t'),$$

where, for any fixed τ', the right-hand side tends to $\bar H'^*(\tau')$ as $t' \to \infty$ because all of the solutions to (4) tend to a globally stable equilibrium. Evaluating this expression at $\tau' = 0$ and using such convergence to an equilibrium of the solutions to (4), we have

$$\frac{H(0,t(t'))}{\varphi_T(t(t'))} = \bar H'(0,t') \longrightarrow \bar H'^*(0) \quad \text{as } t' \to \infty,$$

where we have used that $\tau(0,t') = 0\ \forall t'$. Now, by definition of φ_T it follows that if $H(0,t(t'))$ oscillates periodically, then $\varphi_T(t(t'))$ also oscillates with the same period but with a displacement which is not proportional to that of $H(0,t(t'))$ because F is nonlinear, and a phase shift that depends on T. So, their quotient would not be independent of t' as $t' \to \infty$, and this would contradict its convergence to a stationary value $\bar H'^*(0)$. Therefore, $H(0,t(t'))$ and $\varphi_T(t(t'))$

must tend to constant values when $t' \to \infty$. Moreover, since $0 < F\left(f^m N^0(1-e^{-\xi T})/\xi\right) \leq dt'/dt \leq 1$ (so $t' \to \infty$ implies $t \to \infty$), this convergence implies that $H(0,t)$ tends to a constant value $H^*(0)$ when $t \to \infty$ and, hence, that $\varphi_T(t)$ tends to $F(H^*(0)(1-e^{-\xi T})/\xi)$. So, for any fixed τ, $\varphi_T(t-\tau) \longrightarrow F(H^*(0)(1-e^{-\xi T})/\xi)$ as $t \to \infty$.

Finally, if F_∞ denotes this limit of $\varphi_T(t)$, from (13) it follows that $\lim_{t\to\infty} \tau'(\tau,t) = F_\infty \tau$ and, by (12), we have $\lim_{t\to\infty} H(\tau,t) = F_\infty \bar{H}'^*(F_\infty \tau)$ and $\lim_{t\to\infty} N(\tau,t) = F_\infty \bar{N}'^*(F_\infty \tau)$. Therefore, the solutions of system (11) tend to a unique positive equilibrium $(N^*(\tau_1), H^*(\tau_2))$.

It is interesting to notice that history-dependent rates can induce sustained oscillations in age-dependent models when a positive feedback is assumed. For instance, in [34] the firing rate of a neuron is an increasing function of the global neural activity which, in turn, is an increasing function of the number of neurons that have already fired. In our model, however, there exists a negative feedback between the current activity of burglars and the number of recently committed burglaries thanks to the dissuasive role played by the police, that is, thanks to the presence of the deterrence factor D_T.

3.3. The equilibrium. Once we have seen that the nonnegative solutions of (11) tend to a unique equilibrium, let us study this equilibrium and derive analytical results about the burglary activity. In particular, we are interested in the relation between average quantities like the mean time between consecutive burglaries of the same house and the length T of the observation period.

Following the same ideas as in model (4), fulfilling that the total numbers of burglars and houses are N^0 and H^0 respectively, the equilibrium solution of (11) is implicitly given by

$$N^*(\tau_1) = N^0 \frac{\Pi_b^D(\tau_1)}{\int_0^\infty \Pi_b^D(\tau)\, d\tau}, \quad H^*(\tau_2) = H^0 \frac{\Pi_h^D(\tau_2)}{\int_0^\infty \Pi_h^D(\tau)\, d\tau}, \tag{15}$$

where the probability at equilibrium that a burglar remains inactive up to time τ_1 under the dissuasive action of police measured by D_T is

$$\Pi_b^D(\tau_1) := \exp\left(-D_T(H^*(0)) f_1(\overline{\alpha}_0^*) \int_0^{\tau_1} f_0(\tau)\, d\tau\right), \tag{16}$$

and the probability at equilibrium of a house not being burgled up to time τ_2 also under deterrence is

$$\Pi_h^D(\tau_2) := \exp\left(-\int_0^{\tau_2} D_T(H^*(0))\, \alpha(\tau, N^*(\cdot), \overline{\alpha}_0^*)\, d\tau\right) = \exp\left(-\frac{N^0}{H^0} \frac{\rho_1(\overline{\alpha}_0^*) \int_0^{\tau_2} \rho_0(\tau)\, d\tau}{\int_0^\infty \Pi_b^D(\tau)\, d\tau}\right). \tag{17}$$

For simplicity, in what follows we shall denote the deterrence factor at equilibrium as

$$D_T^* := D_T(H^*(0)) = F\left(H^*(0)(1-e^{-\xi T})/\xi\right). \tag{18}$$

Of course, one expects the action of extra police deployment to reduce criminality. In fact, since $0 < D_T^* \leq 1$, it is clear that the probability of a burglar remaining inactive up to time τ_1 is higher under higher deterrence:

$$\Pi_b^D(\tau_1) = (\Pi_b(\tau_1))^{D_T^*} \geq \Pi_b(\tau_1). \tag{19}$$

Similarly, using the inequality (19) in (17), one can see that also the probability of a house remaining safe up to τ_2 with an increased deterrence effect is, as expected, larger than without it,

$$\varPi_h^D(\tau_2) \geq \varPi_h(\tau_2). \tag{20}$$

Note that, as in the case without active police deployment, the expressions for the equilibrium (N^*, H^*) are not explicit because they both depend on $H^*(\tau)$ through D_T^* and $\overline{\alpha}_0^*$. To compute such equilibrium, first we replace $H^*(0)$ by its formal expression in (15) into the expression of D_T^*,

$$D_T^* = F\left(\frac{H^0(1-e^{-\xi T})}{\xi\int_0^\infty \varPi_h^D(\tau)d\tau}\right),$$

leading to the following equation for D_T^*:

$$D_T^* = F\left(H^0\frac{1-e^{-\xi T}}{\xi}\left(\int_0^\infty \exp\left(-\frac{N^0}{H^0}\frac{\rho_1(\overline{\alpha}_0^*)\int_0^\tau \rho_0(s)\,ds}{\int_0^\infty e^{-D_T^* f_1(\overline{\alpha}_0^*)\int_0^u f_0(s)\,ds}du}\right)d\tau\right)^{-1}\right). \tag{21}$$

Then, along the same lines as we did in the previous section, we obtain the equation satisfied by $\overline{\alpha}_0^*$ that now reads

$$\overline{\alpha}_0^* = \frac{\displaystyle\int_0^\infty \alpha_0(\tau_2)\exp\left(-\frac{N^0}{H^0}\frac{\rho_1(\overline{\alpha}_0^*)\int_0^{\tau_2}\rho_0(\tau)\,d\tau}{\int_0^\infty e^{-D_T^* f_1(\overline{\alpha}_0^*)\int_0^{\tau_1} f_0(\tau)\,d\tau}d\tau}\right)d\tau_2}{\displaystyle\int_0^\infty \exp\left(-\frac{N^0}{H^0}\frac{\rho_1(\overline{\alpha}_0^*)\int_0^{\tau_2}\rho_0(\tau)\,d\tau}{\int_0^\infty e^{-D_T^* f_1(\overline{\alpha}_0^*)\int_0^{\tau_1} f_0(\tau)\,d\tau}d\tau_1}\right)d\tau_2}. \tag{22}$$

The existence of a unique solution $(D_T^*, \overline{\alpha}_0^*)$ of system (21)–(22) is guaranteed because of the convergence of solutions to a globally stable equilibrium. So, we can compute it numerically and, then, the equilibrium densities $(N^*(\tau_1), H^*(\tau_2))$. There is, however, one limiting case where the previous system has a solution with an explicit value of D_T^*, namely, when $T = 0$ because then $D_T^* = 1$. This case corresponds to a model where, roughly speaking, the police does not have any memory of the past committed burglaries and, hence, there is no extra deployment of police resources, which corresponds to the scenario of system (4).

In general, from (18) it follows that the deterrence factor at equilibrium D_T^* is monotonically decreasing in T. That is to say, if the length T of the observation period increases, the police will take into account a larger number of burglaries and, therefore, the deployment of extra police resources will be larger. However, such a reduction of D_T^* has a strictly positive lower bound when $T \to \infty$ due to the presence of the discount term. Precisely, from (15) and (20) we have $H^*(0) \leq H^0/\int_0^\infty \varPi_b(\tau)d\tau$ $\forall T > 0$, and, from (18), it follows that $D_T^* > F\left(H^0/(\xi\int_0^\infty \varPi_b(\tau)d\tau)\right)$ $\forall T > 0$ because F is a decreasing function. Therefore, at some point, considering larger observation periods will not produce a noticeable improvement in the results. Indeed, one can compute the limit values of D_T^* and $\overline{\alpha}_0^*$ as $T \to \infty$ by solving the limit system of equations given by

$$D_T^* = F\left(H^0\frac{1}{\xi}\left(\int_0^\infty \exp\left(-\frac{N^0}{H^0}\frac{\rho_1(\overline{\alpha}_0^*)\int_0^\tau \rho_0(s)\,ds}{\int_0^\infty e^{-D_T^* f_1(\overline{\alpha}_0^*)\int_0^u f_0(s)\,ds}\,du}\right)d\tau\right)^{-1}\right) \tag{23}$$

along with (22).

Under the hypothesis that $\liminf_{\tau\to\infty} f(\tau,\overline{\alpha}_0) > 0\ \forall\overline{\alpha}_0 > 0$ and using (16), the expected time between two consecutive offenses committed by the same burglar under the presence of extra police deployment is given by

$$R_b := \int_0^\infty \tau D_T^* f(\tau,\overline{\alpha}_0^*)\varPi_b^D(\tau)\,d\tau = \int_0^\infty \varPi_b^D(\tau)\,d\tau \geq \int_0^\infty \varPi_b(\tau)\,d\tau, \tag{24}$$

where the inequality follows from (19). Similarly, from the strict positivity of $\alpha_0(\tau)$ and using (20), it follows that the expected time at equilibrium between two consecutive burglaries of the same house with police deterrence is

$$R_h := \int_0^\infty \tau D_T^*\,\alpha(\tau,N^*(\cdot),\overline{\alpha}_0^*)\varPi_h^D(\tau)\,d\tau = \int_0^\infty \varPi_h^D(\tau)\,d\tau \geq \int_0^\infty \varPi_h(\tau)\,d\tau. \tag{25}$$

These expected times depend on T through D_T^*. In fact, by means of the differentiation chain rule and the fact that D_T^* decreases with T, one easily sees that both expected times are increasing functions of T.

The relationship between R_h and R_b follows upon replacing $\varPi_h^D$ in (25) by its expression (17) which gives

$$R_h = \int_0^\infty \exp\left(-\frac{N^0}{H^0}\frac{\rho_1(\overline{\alpha}_0^*)\int_0^{\tau_2}\rho_0(\tau)\,d\tau}{R_b}\right)d\tau_2. \tag{26}$$

As one would expect, the relation is increasing and reflects that the longer a burglar waits to commit the following burglary, the longer a house remains safe. Moreover, in spite of the fact that we are considering deterrence, this relationship turns out to be independent of the deterrence factor D_T^*, and it is the same as the one we have seen in section 2 (see (8)).

Note that, for $\rho(\tau,\overline{\alpha}_0^*) = \rho(\overline{\alpha}_0^*)$, the previous relationship becomes linear and, after integrating, it reads

$$R_h = \frac{1}{\rho(\overline{\alpha}_0^*)}\frac{H^0}{N^0}R_b,$$

which has as proportionality constant the ratio (9) for the constant deterrence case. From this expression it also follows that D_T^* and, hence, R_b are independent of H^0.

4. A numerical example.

In this section, we will consider the deterrence factor given by

$$D_T\left(H\left(0,\cdot\right),t\right) := \exp\left(-\delta\int_{t-T}^t H(0,s)e^{-\xi(t-s)}\,ds\right), \tag{27}$$

which clearly satisfies the hypotheses on F that we have assumed in the previous section. We can think of $\delta > 0$ as a measure of the police reaction per burglary. In this section we will take

$$\delta = 10^{-3}.$$

Moreover, we will consider

$$f_0(\tau) = \frac{\tau}{1+\tau}, \quad f_1(\overline{\alpha}_0) = 1 - e^{-5\overline{\alpha}_0}, \quad \rho_0(\tau) = \alpha_0(\tau), \quad \text{and} \quad \rho_1(\overline{\alpha}_0) = 1$$

with

$$\alpha_0(\tau) = \frac{\tau^3}{10^3 + \tau^3}.$$

The expression for $f_0(\tau)$ reflects a progressive increase of the propensity of burglars to commit an offense. The expression for $f_1(\overline{\alpha}_0)$ assumes that the effect of the mean vulnerability on the recurrence rate is noticeable only when the former is small.

Following the ideas in section 2 for ρ and η, the expressions for $\rho_0(\tau)$ and $\rho_1(\overline{\alpha}_0)$ amount to a proportionality factor that can be written as $\rho(\tau, \overline{\alpha}_0) = \frac{\alpha_0(\tau_2)}{\overline{\alpha}_0}\,\overline{\alpha}_0$, which corresponds to a burglars' allocation function of the form $\eta(\alpha_0(\tau_2), t) = (\alpha_0(\tau_2) + a)/(\overline{\alpha}_0(t) + a)$ with $a = 0$. This simple expression for $\rho(\tau, \overline{\alpha}_0)$ leads to a co-offending factor $\overline{\rho}(\overline{\alpha}_0) = \overline{\alpha}_0$ which, as expected, increases with the mean vulnerability and satisfies $0 < \overline{\rho} \leq 1$. Finally, according to the expression for $\alpha_0(\tau)$, there is an abrupt change in the vulnerability of houses as their age approaches 10, passing from very small vulnerabilities to values close to 1 as τ crosses 10. To model the evolution of burglaries for a particular town one should first adjust the functions and the parameters of the equations to fit with real data not only at a qualitative level but also quantitatively (some data can be found in the Study Group report [31]).

To integrate (11) numerically, we have adapted the explicit numerical scheme for nonlinear models of age-structured population dynamics introduced in [2]. Briefly, this method integrates a first-order PDE along characteristics using the composite midpoint rule for time integration and a second-order modified composite trapezoidal rule for the integration with respect to age. The step size of the grid points over which we integrate the equations is $h = 0.0005$ (discretization parameter). The equilibrium densities have been computed using (7) after solving system (21)–(22) for D_T^* and $\overline{\alpha}_0^*$ with the numeric solver *vpasolve* in MATLAB (R2017a), with all the integrals appearing in the system computed symbolically. We also compute the limit of these quantities by solving the system of equations (22)–(23) with the same procedure.

Figure 1 is a snapshot (at $t = 120$) of the time evolution of $H(\tau, t)$ for $T = 4$, an initial history $h_0(t) \equiv 0$, and a linear decreasing initial condition $(N_0(\tau), H_0(\tau))$ defined on $\tau \in [0, 80]$ and such that $N^0 = 500$ and $H^0 = 10^4$. This figure clearly shows how the left part of the curve (before the jump discontinuity) approaches the equilibrium which has a quite constant profile for $\tau < 10$ because of the extremely low vulnerability of the houses in this range of victimization ages. The right part corresponds to the density of houses that have not been burgled since $t = 0$, that is, the shift of the linear initial condition $H_0(\tau)$ weighted by the probability of still not being burgled. Observe the change of the slope of the profile around $\tau = 130$ which is a trace of the abrupt change in the houses' victimization rate α as a function of τ_2. The jump discontinuity appears because the initial condition is not compatible with the boundary condition data: the value of $H(0, 0)$ computed from the evaluation of the boundary condition at $t = 0$ using the initial condition $(N_0(\tau), H_0(\tau))$ and the initial history $h_0(t)$, $t \in [-T, 0)$, is different from $H_0(0)$.

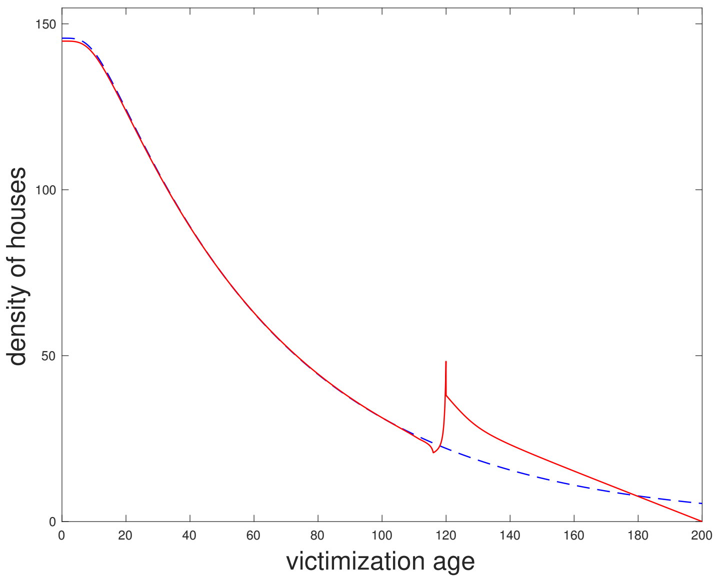

Figure 1. *Snapshot of the time evolution of the density of burglarized houses at $t = 120$ showing the shift of the jump discontinuity arising at $t = 0$ from the incompatibility between the initial condition and the boundary condition. Dashed line: density of houses at equilibrium shown for $\tau \in [0, 200]$. Parameters: $N^0 = 500$, $H^0 = 10^4$, $\xi = 0.1$, $\delta = 0.001$, $T = 4$. For details about the rates, see the main text.*

Now we focus on the dependence of the deterrence factor D_T^*, and both expected times at equilibrium R_b and R_h with respect to different values of N^0 and H^0 for different discount rates ξ.

Figure 2 shows an example of the solution D_T^* of the system (21)–(22) and their limit for a particular set of N^0 and H^0 and for two different values of ξ. On one hand, one observes the monotonicity of D_T^* with respect to T. Also, one can see that the positive horizontal limit as $T \to \infty$ is achieved relatively fast. On the other hand, the effect of decreasing the discount rate ξ is observed to be that of decreasing the deterrence factor D_T^*, but also in that this decrease takes place much more slowly. This validates the consistency of the modified model (11). Finally, we note that, for a given observation period T, one could think of ξ as a tuning parameter to achieve a certain desired reduction on D_T^*. One could also do the opposite, fix the value of the discount rate ξ and decide the optimal observation time T.

Figures 3 and 4 show the expected times as a function of the length T of the observation period and illustrate, for two different values of the discount rate ξ, their dependence on the total number of burglars N^0 (Figure 3) and N^0 and the total number of houses H^0 (Figure 4). These expected times are also a good measure to see if the extra deployment of the police has the desired dissuasive effects. Expressions (24) and (25) show that both deterred expected times increase as more police resources are deployed, as is naturally expected. This happens, for instance, when the length T of the observation period increases or the discount rate ξ decreases (see Figures 3 and 4). The bounded effect of the police deployment in both expected

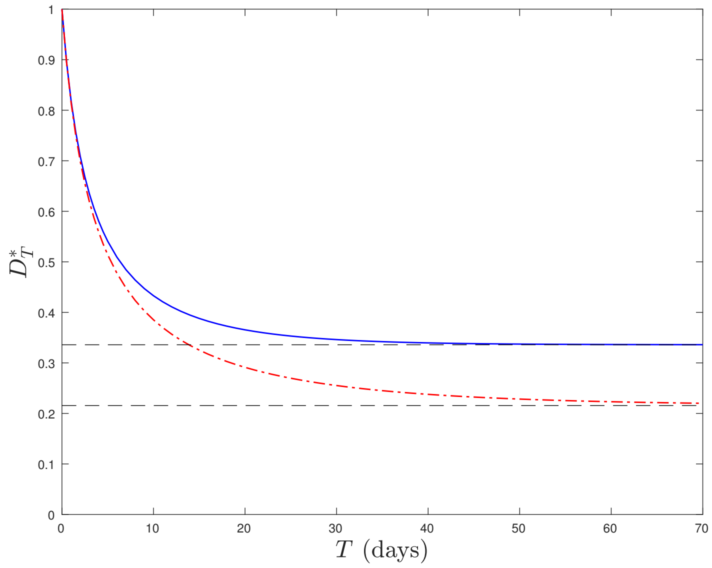

Figure 2. *Deterrence factor at equilibrium, D_T^*, as the solution of the system (21)–(22) when varying the observation period T for two values of the discount rate ξ and for fixed $N^0 = 500$ and $H^0 = 10^5$. Blue solid curve: $\xi = 0.1$; red dash-dotted curve: $\xi = 0.05$. We can observe the decreasing behavior of D_T^* with T, its increasing behavior with ξ, and its limit as $T \to \infty$.*

times as $T \to \infty$ can also be seen in these figures.

Also, in Figure 3 we can see that if the number of burglars in the system N^0 increases, R_b also increases. This means that, as expected, each burglar has to wait longer to commit the next offense if more burglars are present in the model. As the total number of houses H^0 has a very little effect in R_b no graphic of this dependence is included. However, comparing the upper and lower panels of Figure 4 one can see that if the number of burglars in the system N^0 increases, the expected time between two consecutive offenses in the same house R_h decreases. On the other hand, increasing the total number of houses H^0 also increases the deterred expected time R_h, as can be seen comparing the left and right panels of Figure 4. Both dependencies are the ones one would expect.

Figure 5 illustrates the dependence between these two expected times given in expression (26). More concretely, in the top panel we see R_h as an increasing function of R_b, but it is not linear in spite of what it may seem. This is shown in the bottom panel of Figure 5, where the ratio $\frac{R_h}{R_b}$ is plotted. There we can observe that this relation tends to be linear as $R_b \to \infty$.

Finally, in Figure 6 we can see the number of burglaries per day as a function of the police observation length T, that is,

$$H^*(0) = \frac{H^0}{R_h}. \tag{28}$$

As the number of burglaries is a decreasing function of T, we see that the longer the observation period, the higher the deterrence effect of the police, although it always has a positive limit.

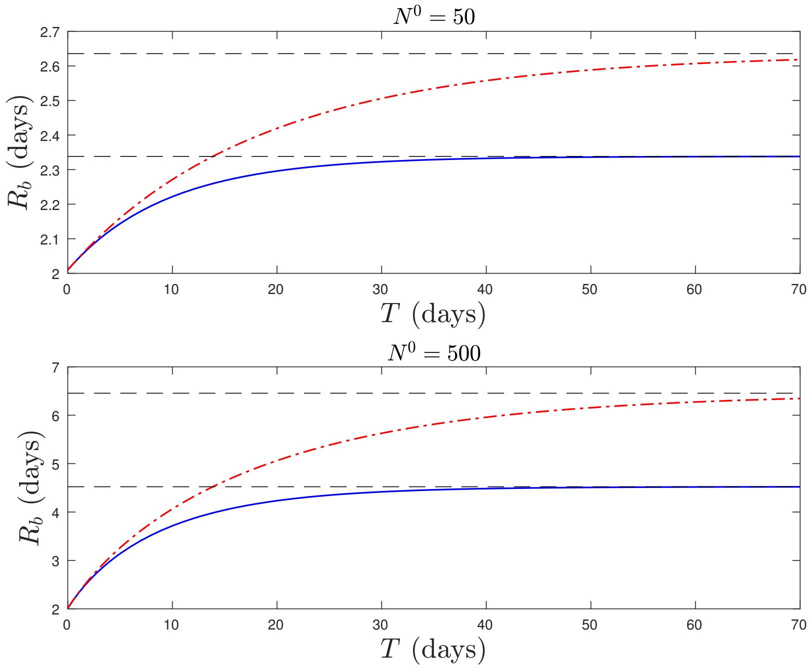

Figure 3. *The burglar expected time R_b given by (24) as a function of the observation period T for $H^0 = 10^5$. Blue solid curve: $\xi = 0.1$; red dash-dotted curve: $\xi = 0.05$. We can observe its increasing behavior with respect to T and N^0, the decreasing behavior with respect to ξ, and the limits as $T \to \infty$.*

In general, it is observed that after some point it is not worth considering longer observation lengths since the decrease in the number of burglaries is unnoticeable. However, by decreasing the value of ξ one sets a new asymptote which also decreases. These two effects are more obvious in cities with a higher initial number of offenses, because the reaction of the police is higher when more burglaries are committed.

From a practical point of view, an interesting issue is to find the observation period that the police should take into account in order to obtain a desired decrease in the burglaries rate. So, in some sense, we could think of this observation period T as a control parameter of the system, by previously fixing the value of ξ.

In conclusion, and to summarize, the more past events are considered by the police (larger T, lower ξ, or larger N^0), the higher the deterrence effect and, hence, the larger the decrease of the number and the percentage of burglaries with respect to the nondeterred model (that is, $T = 0$).

5. Discussion.

We have introduced a model for the dynamics of burglars and victimized houses which takes dynamic police deterrence into account. In contrast to previous works on urban crime modeling, mainly based on the spatio-temporal description of criminal activities, here we have adopted a new approach which focuses on the timing of burglary activity itself, but still introducing heterogeneity in houses and burglars by representing them as population densities with respect to their "ages." The propensity of a burglar to commit an offense is

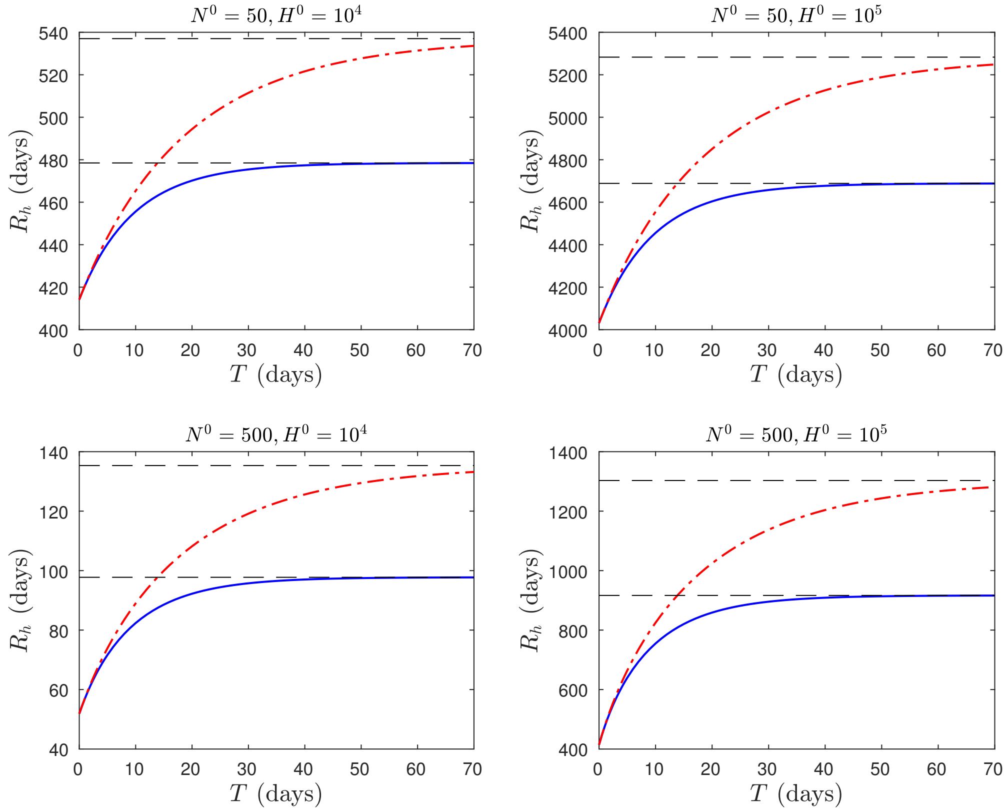

Figure 4. *The house burglaries expected time R_h given by (25) as a function of the observation period T. Blue solid curve: $\xi = 0.1$; red dash-dotted curve: $\xi = 0.05$. Combining the upper and lower panels, we can see its decreasing behavior with respect to N^0. Combining the left and right panels, the increasing behavior with respect to H^0 can be seen. Also, in each panel, we can observe the decreasing behavior with respect to ξ, and the limit of these expected times as $T \to \infty$.*

then a function of the time since its last strike (burglar's age), whereas the vulnerability of a house is a function of the time since it was burgled (victimization age of a house) and it is assumed to be less than or equal to 1. The recurrence rate of a burglar increases with the mean vulnerability of the density of houses, and decreases with the weighted number of burglaries that occurred during the last T time units (observation period). Such a decrease in burglars' activity reflects the behavioral reaction to a greater police deployment occurring after a higher criminal activity (dynamic police deterrence). In turn, the victimization rate of a house is proportional to the number of active burglars per unit of time, and depends on houses' mean vulnerability. In general, such a dependence reflects a decrease of the victimization rate with the level of co-offending among burglars, which is assumed to be a function of houses' mean vulnerability. The simplicity of our model, compared to some of the previous approaches, has allowed us to obtain explicit results (and not only numerical simulations) for the qualitative

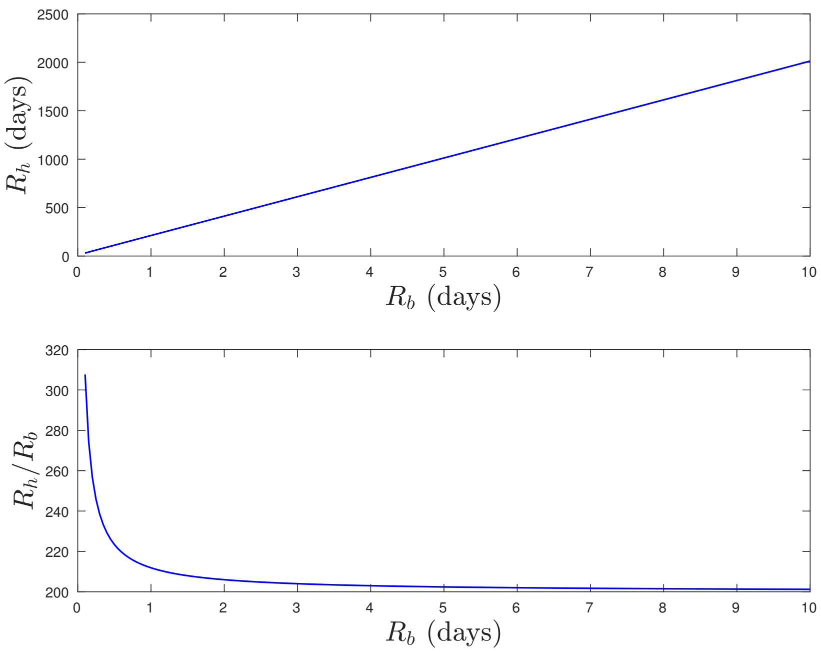

Figure 5. *Top panel: Expected time of consecutive burglaries of a house R_h versus the expected time between offenses committed by a burglar R_b given by (26). Bottom panel: The ratio between these two expected times versus R_b. For the current choice of $\alpha_0(\tau)$, the graph shows that R_h tends to be proportional to R_b for large enough values of the latter. In both panels, $N^0 = 500$ and $H^0 = 10^5$.*

study of the model in terms of different parameters of interest.

The importance of past crimes on the current police deterrence is determined by a discount factor, an exponentially decreasing function of the time elapsed since the occurrence of an offense. Such a discount implies that very long observation periods ($T \to \infty$) will not be of practical use. This is so because, even if the information about all the burglaries in the past were completely available, it is not clear that those occurring a very long time ago could have any relation to the current criminal activity. Such a low impact of burglaries occurring a long time ago on the deterrence factor can be observed in Figure 2 where large values of T do not produce noticeable changes in D^*_T.

It is interesting to observe that by assuming different forms of vulnerability and recurrence rates, several scenarios on the criminal activity can be proposed. This represents novelty with respect to previous models used in the literature. For instance, if one considers vulnerability functions with a high global maximum at small victimization ages, this implies that recently burgled houses are more likely to be burgled again in a short period of time, a concept that has been called *repeat victimization* [39]. On the other hand, an offender's motivation to strike again after committing an offense (e.g., wait until they spend the loot from the last burglary or strike again after a very short period of inactivity) can be introduced into the model by means of suitable dependency of recurrence rate on the burglar's age. The possibility of having

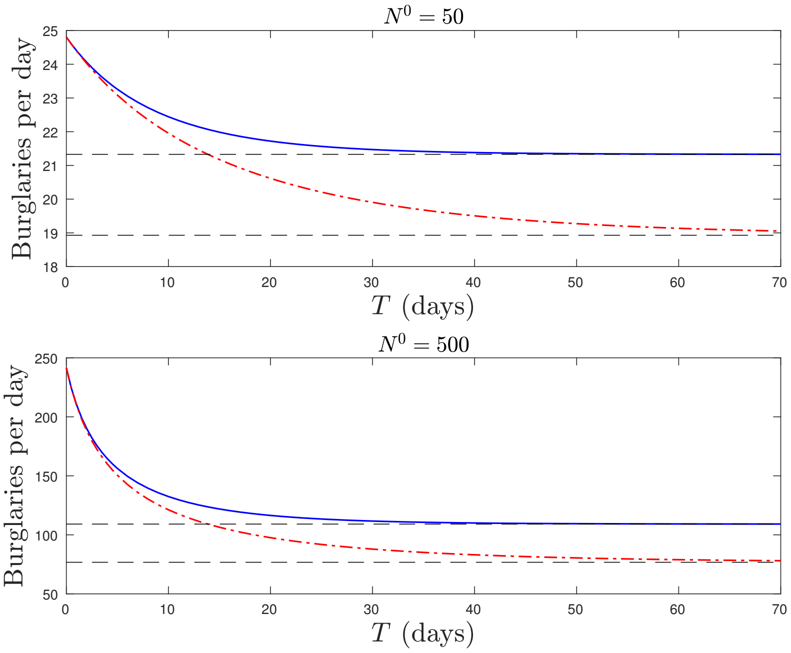

Figure 6. *Rate of burglaries while varying the observation period T (see formula (28)). Blue solid curve: $\xi = 0.1$; red dash-dotted curve: $\xi = 0.05$. For a fixed $H^0 = 10^5$, we can see its decreasing (increasing) behavior with respect to T (ξ), that is, with respect to the deterrence effect of the police, together with the constant limit as $T \to \infty$. Also, we can see that the rate of burglaries increases with N^0.*

different forms of vulnerability and recurrence rates allows us to study, in a future work, the predictions of the model under different scenarios, as well as to test different police strategies in each of them.

From a modeling point of view, one of the main analytic results of the paper is the relationship between the expected time between two consecutive offenses of a burglar, R_b, and the expected time between two consecutive burglaries of the same house, R_h, given by (26). Under the model assumptions, this relationship turns out to be independent of the police deterrence and, moreover, becomes linear when the victimization of a house does not depend on its age.

A police deterrence based upon the number of past burglaries determines the dynamics of the burglar-house system to be history dependent. However, since this deterrence is assumed to be the same for all the burglars regardless of their age, a suitable continuous rescaling of time that depends on the solution itself transforms the original system into one with constant deterrence ($D_T \equiv 1$). This is an interesting result from a mathematical point of view. In the new system, the time variable passes more slowly than in the original one to compensate for the reduction of the burglary rate due to the active police deployment. This transformation, and the fact that the new time tends to infinity when the original one tends to infinity, allow us to prove that the positive solutions of the system with dynamic deterrence tend to a global positive equilibrium. Remarkably, since the transformed system is always the same for any observation period used to compute the police reaction, only the initial condition of the new system and the relationship between both time variables will change with the observation period. Future work will consider an age-dependent impact of a dynamic police deterrence on burglars' activity. This hypothesis introduces more realism into the model but, at the same time, does not allow for such a transformation between models. For instance, under this extended modeling framework, one can assume that the active police deterrence is more effective for burglars that have committed an offense in the past few days than for those who acted a long time ago (because the latter have spent the loot of their last offense).

Appendix A. Asymptotic behavior of the model without police deployment. Introducing the new densities $\bar{N}(\tau,t)$ and $\bar{H}(\tau,t)$ defined in section 2.2 and the time variable $t' = \int_0^t f_1(\overline{\alpha}_0(s))ds$ in system (4) with $f(\tau_1,\overline{\alpha}_0) = f_0(\tau_1)f_1(\overline{\alpha}_0)$ and $\alpha(\tau_2, N(\cdot,t), \overline{\alpha}_0(t))$ given by (3) and (5), we obtain (denoting again by $\bar{N}$ and $\bar{H}$ the new densities as functions of t' and τ')

$$\begin{cases} \partial_{t'}\bar{N}(\tau_1',t') + \partial_{\tau_1'}\bar{N}(\tau_1',t') &= -f_0(\tau_1')\bar{N}(\tau_1',t'),\\ \partial_{t'}\bar{H}(\tau_2',t') + \partial_{\tau_2'}\bar{H}(\tau_2',t') &= -\rho(\tau_2',\overline{\alpha}_0(t'))\displaystyle\int_0^\infty f_0(\tau)\bar{N}(\tau,t')d\tau\,\frac{\bar{H}(\tau_2',t')}{H^0},\\ \bar{N}(0,t') &= \displaystyle\int_0^\infty f_0(\tau)\bar{N}(\tau,t')\,d\tau,\\ \bar{H}(0,t') &= \displaystyle\int_0^\infty f_0(\tau)\bar{N}(\tau,t')\,d\tau\int_0^\infty \rho(\tau,\overline{\alpha}_0(t'))\frac{\bar{H}(\tau,t')}{H^0}\,d\tau, \end{cases}$$

with the corresponding initial conditions given by $\bar{N}(\tau_1',0) = N_0(\tau_1(\tau_1'))/f_1(\overline{\alpha}_0(0))$ and $\bar{H}(\tau_2',0) = H_0(\tau_2(\tau_2'))/f_1(\overline{\alpha}_0(0))$ (recall that $\overline{\alpha}_0(t) = \overline{\alpha}_0(0)\ \forall\, t \le 0$). In particular, it follows that

$$\begin{aligned}\int_0^\infty \bar{N}(\tau',0)\,d\tau' &= \int_{-\infty}^0 \bar{N}(-t_0',0)\,dt_0' = \int_{-\infty}^0 \bar{N}(-t_0'(t_0),0)\,\frac{dt_0'}{dt_0}\,dt_0\\ &= \int_{-\infty}^0 \frac{N(-t_0,0)}{f_1(\overline{\alpha}_0(0))}\,f_1(\overline{\alpha}_0(0))\,dt_0 = \int_0^\infty N(\tau,0)\,d\tau = N^0,\end{aligned}$$

where it is used that $dt'/dt = f_1(\overline{\alpha}_0(0))$ for $t \le 0$, and that $\tau = -t_0$ and $\tau' = -t_0'$ at $t = 0$. Moreover, it is also used that $t_0'(t_0) \to -\infty$ as $t_0 \to -\infty$ because of the strict positivity of $f_1(\overline{\alpha}_0(0))$. Analogous computations show that $\int_0^\infty \bar{H}(\tau',0)\,d\tau' = H^0$. Therefore, the total number of burglars and houses of the transformed initial value problem is the same as in the original one. In fact, using that $\bar{N}(\tau',t')$ and $\bar{H}(\tau',t')$ are functions in $L^1(0,\infty)$, it is immediate to see that the total number of burglars and houses is constant for all time by integrating the corresponding PDEs with respect to τ' from 0 to ∞.

The equation and the boundary condition for $\bar{N}(\tau',t')$ correspond to the well-known linear autonomous age-structured Lotka–McKendrick model (see, for instance, [42, 25]), but in this case the corresponding mortality and fertility rates are equal. Moreover, we note that the equations for $\bar{N}(\tau',t')$ are decoupled from those for $\bar{H}(\tau',t')$. Traditionally, this type of equation has been solved and its asymptotic behavior obtained by integrating them along characteristics and reducing it to a renewal integral equation for the birth rate (which here corresponds to the number of active burglars per unit of time). So, from the renewal theorem with Malthusian parameter equal to zero, it follows that

$$\lim_{t'\to\infty} \bar{N}(\tau', t') = \bar{N}^*(\tau'), \tag{29}$$

pointwise, in $\tau' \geq 0$, where $\bar{N}^*(\tau')$ is the unique equilibrium solution for the density of burglars, with total population N^0 (see, e.g., Chapter 14 in [42]).

Now, using that the solution for the density of burglars $\bar{N}(\tau', t')$ is a known function (e.g., the solution computed from the renewal equation), the victimization rate α given by (3) can be thought of as a function of τ_2 and t. Then, one can reduce the previous system to the following nonlinear nonautonomous first-order PDE:

$$\begin{cases} \partial_{t'} \bar{H}(\tau_2', t') + \partial_{\tau_2'} \bar{H}(\tau_2', t') & = & -\alpha(\tau_2', \overline{\alpha}_0(t'), t')\bar{H}(\tau_2', t'), \\ \bar{H}(0, t') & = & \displaystyle\int_0^\infty \alpha(\tau, \overline{\alpha}_0(t'), t')\bar{H}(\tau, t')\, d\tau, \end{cases} \tag{30}$$

with $\alpha(\tau, \overline{\alpha}_0(t'), t')$ given by (3) replacing $N(\tau, t)$ and $f(\tau_1, \overline{\alpha}_0(t))$ by $\bar{N}(\tau', t')$ and $f_0(\tau_1)$, respectively, computing $\overline{\alpha}_0(t')$ from $\bar{H}(\tau', t')$, and with $\rho(\tau_2, \overline{\alpha}_0) = \rho_0(\tau_2)\rho_1(\overline{\alpha}_0)$.

Now, since ρ satisfies the hypotheses of section 2.2, we can rescale the time variable t' along the same lines as before, namely, by introducing the time variable $t'' = \int_0^{t'} \rho_1(\overline{\alpha}_0(s))ds$. Then, introducing the corresponding new density for the houses, we can transform (30) into a linear nonautonomous PDE. The convergence of $\bar{N}(\tau', t')$ to an equilibrium and the weak ergodicity theorem (see [32, 42]) guarantees that, for any nonnegative initial condition $H_0(\tau)$ and assuming $N_0(\tau)$ continuous, the solution of this linear nonautonomous PDE for the density of houses converges to the unique equilibrium solution with total number of houses equal to H^0.

Finally, from the definition of $\bar{N}$ and $\bar{H}$ at $\tau' = 0$, we have

$$\frac{N(0, t(t'))}{f_1(\overline{\alpha}_0\,(t(t')))} = \bar{N}(0, t') \longrightarrow \bar{N}^*(0) \quad \text{and} \quad \frac{H(0, t(t'))}{f_1(\overline{\alpha}_0\,(t(t')))} = \bar{H}(0, t') \longrightarrow \bar{H}^*(0),$$

as $t' \to \infty$ (recall that $\tau(0, t') = 0\ \forall\, t' \geq 0$). Therefore, since t increases monotonously with t' and tends to ∞ as $t' \to \infty$, $H(0, t)$ and $N(0, t)$ tend to constant values $H^*(0)$ and $N^*(0)$, respectively, as $t \to \infty$. This implies that the solution $(H(\tau, t), N(\tau, t))$ of (4) tends to a unique equilibrium $(H^*(\tau), N^*(\tau))$. Note that if $H(0, t)$ oscillates periodically and $\alpha_0(\tau)$ is not constant (otherwise, $\overline{\alpha}_0(t) = \alpha_0$ and the denominator in both quotients would be constant), then $\overline{\alpha}_0(t)$ and $N(0, t)$ would also be periodic. However, from the relationship $H(0, t) = \overline{\rho}(H(\cdot, t), \overline{\alpha}_0(t))N(0, t)$, it would follow that the previous two quotients cannot both be constant as $t' \to \infty$, thus contradicting the convergence of $\bar{N}(0, t')$ and $\bar{H}(0, t')$ to constant values.

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