The structure of the quiescent core in rigidly rotating spirals in a class of excitable systems

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Abstract

We consider a class of excitable system whose dynamics is described by Fitzhugh-Nagumo (FN) equations. We provide a description for rigidly rotating spirals based on the fact that one of the unknowns develops abrupt jumps in some regions of the space. The core of the spiral is delimited by these regions. The description of the spiral is made using a mixture of asymptotic and rigorous arguments. Several open problems whose rigorous solution would provide insight in the problem are formulated.

1 Introduction, description of the model

There is a wide variety of physical, chemical and biological systems exhibiting spiral wave patterns (cf. [2], [4], [17], [20], [25], [27], [28], [29], [30], [31]). A rough classification of spiral waves would distinguish between the cases in which spiral waves propagate in media which are able to produce oscillations in homogeneous situations and the case in which the oscillation takes place only in the presence of the waves. This second situation arises typically in the propagation of spiral waves in the so-called excitable media. The mathematical theory that describes the waves in these two cases is fundamentally different. Both cases have been studied extensively using numerical and analytical methods. In the case of oscillatory media see for instance [1], [3], [16], [22], [23], [24]. As for excitable media, they have been considered in many papers, some of them are [18], [12], [13], [14], [15], [16], [26].

There exist many reaction-diffusion systems yielding excitable behavior. One of the most commonly studied models in order to gain insight in the behavior of such systems is the Fitzhugh-Nagumo model (cf. [11], [21]):

\[
\varepsilon \frac{\partial u}{\partial t} = \varepsilon^2 \Delta u + f(u,v), \quad \frac{\partial v}{\partial t} = g(u,v), \tag{1.1}
\]

where \(f(u,v)\) and \(g(u,v)\) are usually cubic and linear functions respectively.

Typically, rigidly rotating spiral waves for (1.1) arise for a large class of choices of \(f, g\) satisfying certain properties. The main ones are the existence of a homogeneous stable equilibrium point \((u_s, v_s)\) for the whole system, as well as the fact that the ODE model \(\varepsilon \frac{\partial u}{\partial t} = f(u,v)\) has
two stable steady states $U_-(v) < U_+(v)$ for each value of $v$. Finally $g(U_-(v), v)$, $g(U_+(v), v)$ must have suitable signs in order to give rise to a system that is able to produce oscillations.

The goal of this paper is to suggest an scenario which could explain the appearance of rigidly rotating spirals in systems with the form (1.1), at least for some long time scales. The distinct characteristic of the spirals obtained in this paper is the presence of a quiescent circular core at the center of the spiral where $(u, v)$ remains close to the stable stationary state $(u_s, v_s)$.

The role of the core in spiral waves has been recognized since the earliest researches on the subject (cf. [28]). An analysis of the core of the spiral using asymptotic methods has been made in [13]. The standard folk wisdom about spiral waves assumes that the core should be quiescent because the medium should not have time to recover to the excitable state at such points. In this paper we will obtain examples of nonlinearities $f(u, v)$ and $g(u, v)$ yielding such a behavior for (1.1). We remark, however, that in order to obtain this behavior we must impose some conditions on the shape of the nonlinearities, that are not the usual ones in the studies of excitable media. On the other hand, the idea of understanding the spiral waves splitting the dynamics in different regions has been tackled before (cf. [6], [9], [12], [14], [19]). We will obtain two different sets of appropriate functions. In one of the cases, there will be two steady states for the underlying ODE kinetics, one of them stable and the other unstable. In the second case, there will be one stable steady state for the kinetics associated to the problem, but it will be assumed that the velocity of the planar fronts for constant $v$ will not be a monotone function.

2 Qualitative description of the spiral

In the solution of (1.1) considered in this paper the functions $u, v$ behave in a distinct way in different regions of the plane. We briefly describe in this Section these behaviours as well as the rationale justifying it.

If we take formally the limit $\varepsilon \to 0$ the system (1.1) becomes:

$$0 = f(u, v), \quad \frac{\partial v}{\partial t} = g(u, v).$$

(2.1)

The solution in which we are interested is described in most regions of the space by means of the solutions of (2.1) except for some thin boundary layers where all the terms of (1.1) are needed.

Due to (2.1), for any given value of $v$, $u$ must take the value of one of the steady states.

Stability considerations imply that the only admissible values of $u$ are the stable states for the dynamics

$$\varepsilon \frac{\partial u}{\partial t} = f(u, v),$$

(2.2)

assuming that $v$ is frozen. We will assume in the rest of the paper that there exist exactly two stable steady states of the first equation in (2.2) for any admissible value of $v$, which will be denoted respectively as $U_-(v)$, $U_+(v)$. Therefore, it is natural to look for solutions of (1.1) in which there exist domains $\Omega_-$ and $\Omega_+$ where $u$ takes the values $U_-(v)$, $U_+(v)$ respectively. The traditional terms to denote the state of the points in these two regions are quiescent and excited state respectively. The dynamics of $v$ in these two regions would be governed by the second equation in (2.1).

On the other hand, it will be assumed that the system (2.1) has a trivial stable steady solution, namely $(u, v) = (u_s, v_s)$. In the solution studied in this paper there will exist a region
\( \Omega_s \) where \((u, v) = (u_s, v_s)\). In order to obtain a rigidly rotating spiral we will need to assume that \( \Omega_s \) is a disk.

In our solution the domains \( \Omega_+, \Omega_- \) will be separated by two spiral curves that rotate rigidly with a frequency \( \omega \) about a fixed core \( \Omega_s \). The approximation (2.1) will not be valid near the boundaries separating these three sets \( \Omega_+, \Omega_-, \Omega_s \). The spiral that leaves the quiescent region behind is usually denoted as the \textit{back wave} \( \sigma_b \) and the one moving from the excited region to the quiescent is called the \textit{front wave} \( \sigma_f \).

It is clear that, since according to (2.1), \( u \) must only take one of the two steady values \( U_- \) and \( U_+ \), there will be spatial discontinuities across the borders of the quiescent and excited regions, at least in this leading order approximation given by (2.1). The main novelty of the type of solution considered in this paper is the presence of spatial discontinuities, in the limit as \( \varepsilon \to 0 \), also for the function \( v \). This discontinuity takes place across the boundary separating \( \Omega_s \) from \( \Omega_+ \cup \Omega_- \) which will be denoted by \( \sigma_s \). Notice also that there exist triple points where the three domains \( \Omega_+, \Omega_-, \Omega_s \) meet. In our particular setting there will be two triple points, one of them, denoted by \( x_T \) will lie in the front wave, and the other, denoted by \( x_N \) which will belong to the back front.

The structure of the different regions is depicted in Figure 1.

![Figure 1: Sketch of the core of the spirals](image)

As we shall show in what follows, the nonlinearities \( f(u, v) \) and \( g(u, v) \) should have some particular properties to allow spirals persisting as stationary solutions of (1.1) and such that they maintain this core structure. Furthermore, \( f(u, v) \) must be such that it enables sharp transitions between the inner and the outer parts and also the existence of the above mentioned triple points.
Summary of notation

Excited region: $\Omega_+ = \{ u = U_+(v) \}$, Quiescent region: $\Omega_- = \{ u = U_-(v) \}$,
Core region: $\Omega_s = \{ u = U_s \} = \{ |x| < R \}$,
Front boundary: $\sigma_f$, Back boundary: $\sigma_b$,
Core boundary: $\sigma_s = \partial \Omega_s = \{ |x| = R \}$, $\sigma_s^+ = \sigma_s \cap \partial \Omega_+$, $\sigma_s^- = \sigma_s \cap \partial \Omega_-$,
Outer boundary of the core: $\sigma_o = \partial \Omega_s = \{ |x| = R \}$, $\sigma_s^+ \cap \sigma_s^- = \sigma_s^+ \cap \sigma_s^-$,
Triple points: $x_T = \sigma_s^+ \cap \sigma_f$, $x_N = \sigma_s^- \cap \sigma_b$

Concerning the geometrical configuration of the spiral wave $s$, we will assume that the curves $\sigma_f$, $\sigma_b$ can be parametrized in polar coordinates as

$$\{ \theta = \phi_f (r) : 0 < r < \infty \}, \{ \theta = \phi_b (r) : 0 < r < \infty \}$$

with $\phi_f, \phi_b \in C^1 (R, \infty)$, and such that they do not intersect,

$$\phi_b (r) < \phi_f (r).$$

We may assume by definiteness that

$$\phi_b (R) = 0.$$

In terms of these parametrizations, the two disjoint regions $\Omega_+$, $\Omega_-$ have the boundaries respectively:

$$\partial \Omega_+ = \sigma_f \cup \sigma_b \cup \{ 0 = \phi_b (R) \leq \theta \leq \phi_f (R) \}, \partial \Omega_- = \sigma_f \cup \sigma_b \cup \{ \phi_f (R) \leq \theta \leq 2\pi + \phi_b (R) \}.$$

3 Hypothesis for the energy required to obtain this construction

We note that the model in (1.1) for frozen $v$ is a gradient flow. This gives that the motion of the waves in the different regions can be understood in terms of suitable energy functionals. Let us then define the following potential energy:

$$F (u, v) = - \int_0^u f (\xi, v) \, d\xi.$$ (3.1)

If we freeze $v$, the evolution of (1.1) is just the gradient flow associated to the energy:

$$E [u; v] = \frac{\varepsilon^2}{2} \int (\nabla u)^2 \, dx + \int F (u, v) \, dx.$$ (3.2)

More precisely, $\frac{\partial u}{\partial t}$ would be computed by means of:

$$\varepsilon \left\langle \frac{\partial u}{\partial t}, \varphi \right\rangle = -dE_{[u,v]} (\varphi),$$

where $\varphi \in L^2$ and $\langle \cdot, \cdot \rangle$ is the standard scalar product in $L^2$. The differential is taken only with respect to $u$, assuming that $v$ is fixed.
The function $F(\cdot, v)$ will be assumed to have two local minima for each value of $v$ for the range of values where $v$ moves. These minima will be denoted as $U_-(v)$, $U_+(v)$. We assume:

$$U_-(v) < U_+(v) \quad v \in (v_{\text{min}}, v_{\text{max}}).$$

The values $v_{\text{min}}, v_{\text{max}}$ denote the range of values where $v$ changes. They do not have any other special meaning, they just represent the range of values admissible for $v$. Outside these values the shape of the nonlinearities will not play any role.

The sign of the speed of the wave for a given $v$ can be obtained just by considering the values of the potential energy $F(u, v)$ at the points $u = U_+(v)$, $u = U_-(v)$. This can be seen immediately examining the equation of the travelling waves for the first equation of (1.1) with constant $v$ with the form $u(x, t) = U(\xi)$, $\xi = x - c(v)t$. Then:

$$U_{\xi \xi} + c(v) U_\xi + f(U, v) = 0, \quad \lim_{\xi \to -\infty} U(\xi) = U_-(v), \quad \lim_{\xi \to \infty} U(\xi) = U_+(v),$$

with:

$$c(v) = -\frac{\int_{-\infty}^{\infty} f(U, v) U_\xi d\xi}{\int_{-\infty}^{\infty} U_\xi^2 d\xi} = -\frac{\int_{U_-(v)}^{U_+(v)} f(U, v) dU}{\int_{U_-(v)}^{U_+(v)} U_\xi^2 d\xi}.$$

Notice that we take the following criteria for the speed of the waves. If $U_-(v)$ invades the region previously covered by the state $U_+(v)$ we will say that $c = c(v) > 0$. On the contrary if $U_+(v)$ invades the space previously occupied by $U_-(v)$ we will say that $c(v) < 0$. The rationale behind this criterium is that monotonically increasing waves have positive velocity and monotonically decreasing waves have negative velocity. Therefore:

$$F(U_-(v), v) < F(U_+(v), v) \iff c(v) > 0,$$

$$F(U_-(v), v) > F(U_+(v), v) \iff c(v) < 0,$$

$$F(U_-(v), v) = F(U_+(v), v) \iff c(v) = 0.$$

Notice that the propagation of the wave for fixed $v$ goes always in the direction of decreasing $F(\cdot, v)$ as it could be expected from energy considerations. That is to say, the energy of the wave behind the front is always lower than the energy ahead of it.

### 3.1 List of conditions required for $f(u, v)$, $g(u, v)$.

We will assume always, that $f, g \in C^2(\mathbb{R}^2)$. The conditions that we will require for both functions are restricted to a set of values, but we can always assume by simplicity that they are defined in the whole $\mathbb{R}^2$ by means of suitable extensions that will not affect the dynamics of the considered solutions of (1.1).

(A1) For each $v \in (v_{\text{min}}, v_{\text{max}})$ there exist three roots of the equation $f(u, v) = 0$ for $u \in \mathbb{R}$. They will be denoted as $U_-(v), U_0(v)$ and $U_+(v)$ and they satisfy:

$$U_-(v) < U_0(v) < U_+(v) \quad v \in (v_{\text{min}}, v_{\text{max}}).$$

(A2) The function $F(\cdot, v)$ has a local minimum at the points $U_-(v), U_+(v)$ and a local maximum at $U_0(v)$ for any $v \in (v_{\text{min}}, v_{\text{max}})$.
(A3) There exists a $v^* \in (v_{\text{min}}, v_{\text{max}})$ such that $F(U_-(v^*), v^*) = F(U_+(v^*), v^*)$.

(A4) Under the assumptions (A1)-(A3) the function $c(v)$ defined by means of (3.3) is strictly monotonically increasing for $v \in (v_{\text{min}}, v_{\text{max}})$.

Notice that, (A3) implies there exists a unique $v^* \in (v_{\text{min}}, v_{\text{max}})$ such that $c(v^*) = 0$, and the monotonicity assumption in (A4) implies that $c(v)$ has only this zero for $v \in (v_{\text{min}}, v_{\text{max}})$.

(A5) Suppose that $U_+(v), U_-(v)$ are as in Assumption (A1). Then $g(U_+(v), v) > 0$ for $v \in (v_{\text{min}}, v_{\text{max}})$. On the other hand there exists $\bar{v}_s, v_s \in (v_{\text{min}}, v_{\text{max}})$, $v^* < \bar{v}_s < v_s$ such that $g(U_-(\bar{v}_s), \bar{v}_s) = g(U_-(v_s), v_s) = 0$ and $g(U_-(v), v) < 0$ for $v \in (v_{\text{min}}, v_{\text{max}})$ and $g(U_-(v), v) > 0$ for $\bar{v}_s < v < v_s$. Moreover, for any $v_{\text{min}} < v_1 < v_2 < \bar{v}_s$ the following inequalities hold:

\[ g(U_-(v_1), v_1) < g(U_-(v_2), v_2), \quad g(U_+(v_1), v_1) < g(U_+(v_2), v_2). \]  

(A6) An alternative, weaker condition, that implies (A5) if we are willing to impose differentiability properties in the functions $f$ and $g$ is:

\[ \frac{d}{dv}(g(U_\pm(v), v)) > 0, \]

for $v \in (v^* - \varepsilon_0, v^* + \varepsilon_0) \subset (v_{\text{min}}, v_{\text{max}})$, $\varepsilon_0 > 0$.

A simple example that could satisfy all these conditions would be something of the form,

\[ f(u, v) = (1 - u)(u - h(v))u, \quad g(u, v) = -(v - \sigma)^2 + \Gamma uv + C, \]

taking, for instance, $h(v) \in (0, 1)$ for $v \in (v_{\text{min}}, v_{\text{max}})$ linear and for suitable values of the parameters. In particular it is easy to check that $h(v) = 7/18 + 1/3v$, $\sigma = 1$, $\Gamma = 1$ and $C = 1/4$ would work (see Figure 2 left and subsection §4.3 for details). In this example the quiescent and excited state would simply correspond to $U_- = 0$ and $U_+ = 1$ and the equilibrium points would take place at $\bar{v}_s = 1/2$ and $v_s = 3/2$.

It is important to remark that the structure of functions $(f, g)$ described here does not describe a standard excitatory system due to the fact that there would be two steady states for the system, namely $(U_-(\bar{v}_s), \bar{v}_s)$ and $(U_-(v_s), v_s)$, being the first unstable and the second stable.

There is an alternative way of constructing functions $(f, g)$ with the requirements needed in order to have the mechanism of spiral wave described in this paper. This new approach would require to have only one steady state for the dynamics of the "underlying kinetics". However, the velocity function $c = c(v)$ would not be monotonic for the whole range of values of $v$. In this second set of assumptions we will assume that the value of $v$ associated to the steady state is below $v^*$. This is not really relevant, because the definition of $v$ has a large degree of arbitrariness and it would be possible to reparameterize it in order to reverse the relative position of $v^*$, $v_s$ making the corresponding changes of signs.

Conditions (A1) and (A3) also hold in this second approach. However, the following conditions substitute (A4)-(A6):

(B4) Under the assumptions (A1)-(A3), the function $c(v)$ defined by means of (3.3) is strictly monotonically decreasing for $v \in (v_{\text{min}}, v_{\text{vel}})$, and increasing for $v \in (v_{\text{vel}}, v_{\text{max}})$, being $v_{\text{vel}}$ a local minimum of the velocity. There exists a unique $v^{**} \in (v_{\text{min}}, v_{\text{vel}})$ and a unique $v^* \in (v_{\text{vel}}, v_{\text{max}})$ such that $c(v^*) = c(v^{**}) = 0$. 

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(B5) Suppose that $U_+(v)$, $U_-(v)$ are as in assumption (A1). Then $g(U_+(v), v) > 0$ for $v \in (v_{\min}, v_{\max})$. On the other hand there exists $v_s \in (v_{\min}, v_{\max})$, $v_s < v^*$ such that $g(U_-(v_s), v_s) = 0$ and $g(U_-(v), v) > 0$ for $v \in (v_{\min}, v_s)$ and $g(U_-(v), v) < 0$ for $v \in (v_s, v_{\max})$. Moreover, for any $v_s < v_1 < v_2 < v_{\max}$ the following inequalities hold:

$$g(U_-(v_1), v_1) > g(U_-(v_2), v_2), \quad g(U_+(v_1), v_1) > g(U_+(v_2), v_2).$$  \hspace{1cm} (3.5)

(B6) Alternatively, if we impose differentiability on the functions $f$ and $g$, condition (B5) is automatically satisfied by requiring the weaker condition:

$$\frac{d}{dv} (g(U_-(v), v)) < 0, \quad \frac{d}{dv} (g(U_+(v), v)) < 0,$$

for $v \in (v^* - \varepsilon_0, v^* + \varepsilon_0) \subset (v_{\min}, v_{\max})$, $\varepsilon_0 > 0$.

(B7) The global minimum of $F(\cdot, v_s)$ is reached at $u = U_-(v_s)$.

Notice than an assumption similar to (B7) is not necessary with the first set of conditions (A1)-(A6), due to the fact that $v_s > v^*$ along with $c(v)$ being monotonically increasing. Indeed, note that this condition is equivalent to $c(v^*) > 0$ with the definition of the velocity provided in (3.3).

A model satisfying these conditions is for example given by

$$f(u, v) = u(1-u)(u-h(v)), \quad g(u, v) = u - \gamma v,$$

with a suitable $h(v) \in (0, 1)$ for $v \in (0, 1)$. However, $h(v)$ cannot now be simply linear since this would not give the right monotonicity properties to the velocity. A possibility would be to take $h(v) = 1 + av(v-1)$. See Figure 2 right and section §4.3 for details.

3.2 Description of the spiral waves in the two different cases

The wave consists in a first front, locally described by a traveling wave with $v$ changing continuously and where $u$ jumps from $U_-(v)$ to $U_+(v)$. Our criteria of signs implies that $c(v) < 0$ across such a front.

Notice that along the region close to $\sigma_+^-$ from the outside of the core we have that $v$ is increasing. On the other hand, since the arrival of the first front (where $u$ jumps from $U_-$ to $U_+$) to the end of the region $\partial \{u = U_+(v)\}$ (where there is a contact with the other front where $u$ jumps from $U_+$ to $U_-$), the values of $c$ must increase from $c(v) < 0$ to $c(v) = 0$. (The intersection point of $\sigma$ with the second front is a point where $c = c(v) = 0$). Therefore, since $v$ is increasing and $c$ is increasing, we have that along this region $c = c(v)$ is a monotonically increasing function.

On the other hand, after arriving to the region where $u = U_-(v)$ we have that $g(U_-(v), v) < 0$, so $v$ starts decreasing. Near the boundary $\sigma_+^-$ we have that $c$ decreases from 0 to the value of the velocity at the first transition front (from $U_-$ to $U_+$). Therefore, the function $c(v)$ must be increasing. In other words, we need to require that $c(v)$ is an increasing function for the whole set of values of $v$ outside the core.

It is worth to remark that along the outer boundary of the core, $\sigma_+^-$, the function $c(v)$ takes a maximum at the detachment point of the second front and a minimum at the arrival point of the first front.
Figure 2: Left: nullclines for a system satisfying conditions (A1)-(A6). Right: nullclines for a system satisfying conditions (A1)-(A3), (B4)-B(7). The dotted lines represent in both figures \( v = v_{\text{min}} \) and \( v = v_{\text{max}} \).

The main properties that we need for \( c(v) \) (at least near \( \sigma^+ \)) are:

\[
\begin{align*}
    c(v) \text{ increasing for } v & \in [v_T, v^*] \subset (v_{\text{min}}, v_{\text{max}}), \\
    c(v^*) & = 0.
\end{align*}
\]  

(3.6)  

(3.7)

We denote as \( v_T \) the value of \( v \) at the intersection between the front where \( u \) jumps from \( U^- \) to \( U^+ \) and \( \sigma \) (\( T \) stands for the triple point). The function \( c(v) \) has a minimum in \( \sigma^+ \) at the point \( v = v_T \) and has a maximum at \( v = v^* \), where we have \( c(v^*) = 0 \).

As for the connections of the steady layers, where changing values of \( v \) connect \( v_s \) with the values of \( v \) at \( \sigma_s^0 \), it is convenient to describe the structure of steady states and energies, at least locally for the values of \( v \) in \( \sigma_s^0 \). There exist two branches \( u = U_-(v) \), \( u = U_+(v) \) for some interval \( (v_{\text{min}}, v_{\text{max}}) \) containing \( [v_T, v^*] \). We have that \( c(v) \leq 0 \) for \( v \in [v_T, v^*] \), and hence, \( F(U_-(v), v) > F(U_+(v), v) \) for \( v < v^* \).

It remains now to study the values of the energy and the different functionals at the stationary or stall point \((u_s, v_s)\). We have basically two possibilities:

(i) Suppose that (A1)-(A6) holds, so \( v_s > v^* \). Notice that in this case, \( c(v) \) is monotonically increasing for \( v \in (v_{\text{min}}, v_{\text{max}}) \). Then, we would have \( c(v_s) > 0 \), whence \( U_- \) tends to invade \( U_+ \), at \( v_s \), whence \( F(U_-(v_s), v_s) < F(U_+(v_s), v_s) \). Since \( u_s = U_-(v_s) \) it then follows that \( u_s \) is a global minimum for \( F(\cdot, v_s) \).

In this case we need two types of stationary connections. One of them is the connection between the stable core \( \Omega_s \) and the excited region \( \{u = U_+(v)\} \), where \( F(U_+(v), v) < F(U_-(v), v) \) since \( v < v^* \). Therefore this is a connection between two local minima (with \( v \) non-constant).
The second connection required is the one between \( \Omega_s \) and \( \{ u = U_-(v) \} \). In the region \( \{ u = U_-(v) \} \) near \( \sigma_v^s \) we have also \( F(U_+(v), v) < F(U_-(v), v) \) due to the fact that \( v < v^* \). The connection needed is then between the point \( (u_s, v_s) \) and \( U_-(v), v \) with \( v < v^* \). Since \( U_-(v) \) is not a minimizer, the existence of such a connection is not clear at all. However, such a connection exists for a large class of nonlinearities \( f \), for instance those for which \( U_-(v) \) is constant for \( v \in (v_{\min}, v_{\max}) \). In such a case, the connection is just \( U_-(v) \). For similar potentials we can obtain the existence of such connections too. It is important to take into account that this connection is not a minimizer, but a metastable solution. However, this is sufficient for the argument. Notice that the metastable solution is reached as a consequence of the own evolution of the system.

(ii) The second possibility is to assume (A1)-(A3), (B4)-(B7), so \( v_s < v^* \). In this case, the no monotonicity assumed for \( c(v) \) in (B4) could look strange at a first glance. The reason for this assumption is due to the fact that we need to have a global minimizer for \( F(\cdot, v_s) \) at \( u = U_-(v_s) = u_s \). This is needed to obtain the connecting layers between the different regions. If we had \( c(v) < 0 \) for any \( v < v^* \) we would obtain that \( F(U_-(v_s), v_s) > F(U_+(v_s), v_s) \). However, this is not compatible with \( F(\cdot, v_s) \) having a global minimizer at \( u = u_s \). Therefore, if \( v_s < v^* \) we need to have an additional value of \( v \) where \( c(v) \) vanishes, and this explains the meaning of assumption (B4).

In any case, assumptions (B4) and (B7) ensure that it is possible to obtain all the required connections by means of variational arguments.

Let us remark that outside the core \( \Omega_s \), in each of the fronts, the sign of \( c(v) \) remains constant. In the first front (transition \( U_- \rightarrow U_+ \)) we have \( c < 0 \). In the second front (transition \( U_+ \rightarrow U_- \)) we have \( c > 0 \). Notice that there are two lines starting at the point in \( \sigma_v^s \) where \( c = 0 \), that enter respectively into the regions \( \{ u = U_+(v) \} \) and \( \{ u = U_-(v) \} \). These lines are interesting because they separate regions where \( u \) is at a global minimum of the potential energy \( F \) and regions where \( u \) is at a local minimum. At the line where \( c = 0 \) we have that \( u \) continues taking the value \( u = U_+ \) but the state is metastable. Then there is a front where \( u \) jumps to \( U_- \) and this is a global minimizer. Finally, there is another line with \( c = 0 \) in the domain \( \{ u = U_-(v) \} \) where \( u \) becomes metastable again, and this is followed by the first front jumping to steady state, closing the loop.

4 Analysis of the spiral waves outside the core region under the sharp interface approximation.

We first derive an approximation for the solutions of (1.1) for the region outside the core as \( \varepsilon \rightarrow 0 \). Basically we just replace the system (1.1) by a sharp interface model, something that is customary in the analysis of spiral waves in excitable media (cf. for instance [12]).

4.1 Derivation of the equations for the spiral fronts driving the spiral waves outside the core: Sharp interface approximation.

We now formulate a system of equations that are motivated by the original FN system (1.1) in the limit \( \varepsilon \rightarrow 0 \). Since the dynamics of the \( u \) variable is much faster than the dynamics associated to \( v \), it is natural to suppose that, in the limit \( \varepsilon \rightarrow 0 \), the plane \( \mathbb{R}^2 \) is divided for any \( t > 0 \) in a set of open regions where the solution \( u \) take only the values \( U_+(v), U_-(v) \)
in some open sets. The rest of the space would consist in the boundaries of such regions. We will denote such regions from \( \mathbb{R}^2 \), which are changing in time, as \( \mathcal{U}_+ (t) \), \( \mathcal{U}_- (t) \). The resulting model that we will obtain will be then an evolution problem for the domains \( \mathcal{U}_+ (t) \), \( \mathcal{U}_- (t) \) and the function \( v (x,t) \).

The main difference between the approach considered in this paper and the usual one, for instance in [12], [13] and [26] is that we will allow the function \( v \) to have discontinuities, along specific curves in the limit \( \varepsilon \to 0 \). Notice then that there would be many possible interfaces separating different regions, as well as different types of triple points where these interfaces intersect. More precisely we can have interfaces contained in each of the regions \( \mathcal{U}_+ (t) \) and \( \mathcal{U}_- (t) \) along which \( v \) is discontinuous. We can have also interfaces separating the regions \( \mathcal{U}_+ (t) \), \( \mathcal{U}_- (t) \). It is not difficult to imagine more involved types of boundaries and discontinuity points. However, our goal in this paper is not to consider the most general evolution problem. Therefore, we will restrict our attention to a very specific type of geometry.

We will suppose that the domain \( \mathcal{U}_- (t) \) is the union of \( \Omega_\varepsilon \) and the domain \( \mathcal{R}_\alpha (\Omega_-) \), where \( \mathcal{R}_\alpha \) is the rotation of angle \( \alpha \) centered at the origin. Notice that since \( \Omega_\varepsilon \) is a disc, \( \mathcal{R}_\alpha (\Omega_\varepsilon) = \Omega_\varepsilon \). On the other hand, we will assume that the domain \( \mathcal{U}_+ (t) \) is just \( \mathcal{R}_\omega (\Omega_+) \). We will assume also that the function \( v \) is stationary in a coordinate system rotating with angular velocity \( \omega \). Then, it will have the form \( v = v(r, \theta - \omega t) \equiv v(r, \phi) \). In other words, the whole structure of domains and functions is rotating around the origin without deformation at constant angular velocity \( \omega \).

We first begin looking for spiral waves for the following model. We will assume that the interfaces separating \( \mathcal{U}_+ (t) \) and \( \mathcal{U}_- (t) \), where \( v \) is a continuous function, evolve according to the equation:

\[
V_n = -c(v),
\]

where \( n \) will denote the outer normal to \( \mathcal{U}_+ (t) \). The function \( c(v) \) is the speed of the one-dimensional fronts that has been obtained at the beginning of Section 3. Systems similar to (4.1) have been extensively used studying limits of FN models (see for instance [13], [26] or [28] among others). Equation (4.1) can be formally justified assuming that the limit interfaces separating regions are smooth, because in such a case the thin layers separating regions where \( u \) is close to \( U_+ (v) \) and \( u \) is close to \( U_- (v) \) can be described by means of the one-dimensional fronts in Section 3. These fronts have been studied rigorously for some excitable systems, for instance, in [7] or [8].

We will also impose that the function \( v \) satisfies the following set of differential equations:

\[
\frac{\partial v}{\partial t} = g(U_+ (v), v) \text{ in } \mathcal{U}_+ (t), \quad \frac{\partial v}{\partial t} = g(U_- (v), v) \text{ in } \mathcal{U}_- (t),
\]

\( v \) is continuous in \( \mathbb{R}^2 \setminus (\Omega_\varepsilon) \).

So far we will not make any particular assumption concerning the constraints that the domains \( \Omega_+, \Omega_- \), the fronts \( \sigma_b, \sigma_f \) and the function \( v \) must satisfy at the regions of discontinuity of \( v \), namely \( \partial \Omega_\varepsilon \). Of course, we cannot expect that the limits of FN could give arbitrarily domains at such discontinuities. As a first step we will just study the solutions of (4.1), (4.2). However, we will describe later some of the constraints that must be imposed for these solutions arising from the fact that we are interested in describing limits of the FN model (1.1).

We now reformulate (4.1), (4.2) as differential equations for the parametrizations of \( \sigma_b, \sigma_f \) (cf. (2.3)). Due to our geometrical assumptions on the domains, the interfaces separating
\( \mathcal{U}_+ (t) \) and \( \mathcal{U}_- (t) \) contained in \( \mathbb{R}^2 \setminus (\Omega_s) \) are made of two pieces, namely \( \mathcal{R}_{\omega t} (\sigma_b) \), \( \mathcal{R}_{\omega t} (\sigma_f) \). The normal vector \( n \) to these rotating fronts is given by:

\[
\begin{align*}
n &= \mathcal{R}_{\omega t} \left( \frac{r \phi'_b (r) e_r - e_\theta}{\sqrt{1 + r^2 (\phi'_b (r))^2}} \right) \text{ at } \mathcal{R}_{\omega t} (\sigma_b), \\
n &= \mathcal{R}_{\omega t} \left( \frac{-r \phi'_f (r) e_r - e_\theta}{\sqrt{1 + r^2 (\phi'_f (r))^2}} \right) \text{ at } \mathcal{R}_{\omega t} (\sigma_f),
\end{align*}
\]

where \( e_r, e_\theta \) are unit vectors in the radial and azimuthal direction respectively. We assume that \( e_\theta \) points in the direction of increasing \( \theta \). The velocity vector of the points of \( \mathcal{R}_{\omega t} (\sigma_b) \), \( \mathcal{R}_{\omega t} (\sigma_f) \), assuming that they are parametrized by \( \sigma_b, \sigma_f \), is \( \omega \mathcal{R}_{\omega t} (\sigma_b) e_\theta \) and \( \omega \mathcal{R}_{\omega t} (\sigma_f) e_\theta \) respectively. Therefore,

\[
\begin{align*}
r \omega \sqrt{1 + r^2 (\phi'_b (r))^2} &= c(v_b) \quad \text{for } \sigma_b, \\
r \omega \sqrt{1 + r^2 (\phi'_f (r))^2} &= -c(v_f) \quad \text{for } \sigma_f,
\end{align*}
\]

where from now on we denote as \( v_b = v_b (r) \) and \( v_f = v_f (r) \) the values of the function \( v \) at \( \sigma_b, \sigma_f \) respectively.

On the other hand, the fact that \( v \) is stationary in a rotating system of coordinates implies, using (4.2) that:

\[
-\omega \frac{\partial v}{\partial \phi} = g(U_+ (v), v) \quad \text{in } \Omega_+.
\]

The original system of equations (4.1), (4.2) has then been reduced in the rotating reference system to (4.3)-(4.5). We can further simplify (4.5) as follows. Suppose that Assumptions (A1)-(A6) hold. In the region \( \Omega_s \) we have:

\[
-\omega \frac{\partial v}{\partial \phi} = g(U_- (v), v) \quad \text{for } \phi \in [0, 2\pi], \quad v (\phi + 2\pi) = v (\phi).
\]

The regularity properties assumed for \( f, g \) imply that the only periodic solutions of the first order differential equation in (4.6) are the constant ones. Due to Assumption (A5) it then follows that \( v = v_s \) in \( \Omega_s \).

Concerning the region \( \mathbb{R}^2 \setminus (\Omega_s) \) we need to impose that the function \( v \) is single-valued, i.e. \( v (\phi + 2\pi) = v (\phi) \). Integrating (4.5) in the regions \( \Omega_+, \Omega_- \) respectively we obtain:

\[
\begin{align*}
\int_{v_f (r)}^{v_b (r)} \frac{dv}{g(U_+ (v), v)} &= \frac{\phi_f - \phi_b}{\omega} \quad \text{in } \Omega_+, \\
\int_{v_f (r)}^{v_b (r)} \frac{dv}{g(U_- (v), v)} &= -\frac{(2\pi + \phi_b) - \phi_f}{\omega} \quad \text{in } \Omega_-,
\end{align*}
\]

where \( \phi_b (r) \) and \( \phi_f (r) \) are as in (2.3). We have used (4.5) that combined with Assumption (A5) yields also \( v_f (r) < v_b (r) \) for \( r \geq R \).
Subtracting equations (4.7) and (4.8) one gets

\[ \int_{v_f(r)}^{v_b(r)} \frac{dv}{g(U_+(v), v)} - \int_{v_f(r)}^{v_b(r)} \frac{dv}{g(U_-(v), v)} = \frac{2\pi}{\omega}. \]  

(4.9)

We define the functions:

\[ G_+(v) = \int_v^v \frac{1}{g(U_+(s), s)} ds, \]
\[ G(v) = \int_v^v \left[ \frac{1}{g(U_+(s), s)} - \frac{1}{g(U_-(s), s)} \right] ds. \]

Then (4.7), (4.9) imply

\[ G_+(v_b(r)) - G_+(v_f(r)) = \frac{\phi_f - \phi_b}{\omega}, \]
\[ G(v_b(r)) - G(v_f(r)) = 2\pi/\omega. \]

(4.10)

(4.11)

Notice that:

\[ G'_+(v) = \frac{1}{g(U_+(v), v)} - \frac{1}{g(U_-(v), v)} > 0, \]
\[ G'_+(v) < G'_+(v). \]

(4.12)

(4.13)

Taking into account that the three equations (4.10), (4.8) and (4.11) are linearly dependent, we have that the original problem for the domains \( \mathcal{U}_+ \), \( \mathcal{U}_- \) and the function \( v \) may be reduced to the system of equations (4.3), (4.4), (4.10) and (4.11).

It remains to prescribe initial values for \( \phi_b(R), \phi_f(R), v_b(R), v_f(R) \). We have already assumed, using the invariance of the problem under rotations, that \( \phi_b(R) = 0 \) (cf. (2.5)). On the other hand, the monotonicity of \( G(v) \) (cf. (4.12)) combined with (4.11) provides \( v_f(R) \) in terms of \( v_b(R) \). Finally, (4.10) at \( r = R \) gives \( \phi_f(R) \). Notice that (4.12), (4.13) imply that \( \phi_f(R) < 2\pi \). Therefore, there is only one free constant left in order to determine the solutions of (4.3), (4.4), (4.10) and (4.11) under the assumption (2.5) and we can assume that such free number is \( v_b(R) \).

Kinematic considerations as well as the fact that \( \omega > 0 \) show that \( c(v_f(R)) \leq 0 \) and \( c(v_b(R)) \geq 0 \). Then, Assumption (A4) implies that

\[ v_f(R) \leq v^* \leq v_b(R). \]  

(4.14)

Notice that (4.11) and (4.12) imply the strict inequality \( v_f(R) < v_b(R) \). Then, the sign \( \leq \) is \( < \) in at least one of the two inequalities in (4.14).

Let us denote as \( \alpha_b \) and \( \alpha_f \) the angles of the tangent vectors to \( \sigma_b \) and \( \sigma_f \) at \( \partial \Omega \), respectively. A geometrical argument shows that:

\[ R \omega \sin(\alpha_b) = c(v_b(R)), \quad R \omega \sin(\alpha_f) = c(v_f(R)). \]  

(4.15)

Notice in particular that \( v_b(R) = v^* \) if and only if \( \alpha_b = 0 \) and \( v_f(R) = v^* \) if and only if \( \alpha_f = 0 \).
4.2 On the existence of spiral waves for the sharp interface approximation model.

The main result of this Subsection is the following:

**Theorem 1** Suppose that either the assumptions (A1)-(A6) or (A1)-(A3), (B4)-(B7) are satisfied. Then, for any \( R > 0 \) there exists \( \omega_0 = \omega_0(R) > 0 \) such that for any \( \omega \geq \omega_0 \) and \( v_{f,0} \in (v_{\min}, v^*) \) such that \( v_{b,0} \equiv G^{-1}(G(v_{f,0}) + 2\pi/\omega) \geq v^* \) there exists a unique solution of (4.3), (4.4), (4.10) and (4.11) \((\phi_b, \phi_f, v_b, v_f)\) such that \( \phi_b(R) = 0 \), \( v_f(R) = v_{b,0}, v_b(R) \) is given by \( v_b(R) = v_{b,0} \) and \( \phi_f(R) = \omega [G_+(v_{b,0}) - G_+(v_{f,0})] \). Moreover \( \phi_b(r) < \phi_f(r) \) for \( r \in [R, \infty) \).

**Proof.** Suppose first that there exists a solution of (4.3), (4.4), (4.10) and (4.11) with the properties stated in the Theorem. Equations (4.3), (4.4) yield:

\[
\phi'_b = \pm \frac{1}{r} \sqrt{\frac{r \omega}{c(v_b)}}^2 - 1, \quad r > R, \tag{4.16}
\]

\[
\phi'_f = \pm \frac{1}{r} \sqrt{\frac{r \omega}{c(v_f)}}^2 - 1, \quad r > R, \tag{4.17}
\]

where a priori both choices of the signs are possible in each of the equations. It turns out that the choice of signs in (4.16), (4.17) must be a very specific one. Using (4.11) and (4.12) we can write

\[
v_b = G^{-1} \left( G(v_f) + \frac{2\pi}{\omega} \right), \tag{4.18}
\]

for \( v_f \in (v^* - \delta, v^*) \) if \( \delta > 0 \) is sufficiently small and \( \omega \) is sufficiently large.

On the other hand, differentiating (4.10) we obtain:

\[
\frac{v'_b}{g_+(v_b)} - \frac{v'_f}{g_+(v_f)} = \frac{\phi'_f - \phi'_b}{\omega}, \tag{4.19}
\]

where:

\[
g_+(v) = g(U_+(v), v), \quad g_-(v) = g(U_-(v), v).
\]

Differentiating (4.11) we obtain:

\[
\left( \frac{1}{g_+(v_b)} - \frac{1}{g_-(v_b)} \right) v'_b = \left( \frac{1}{g_+(v_f)} - \frac{1}{g_-(v_f)} \right) v'_f, \tag{4.20}
\]

whence, using this formula into (4.19):

\[
v'_f = H(v_f) (\phi'_f - \phi'_b), \tag{4.21}
\]

where \( \phi'_f, \phi'_b \) must be computed using (4.16), (4.17). The function \( H \) is defined by means of:

\[
H(v_f) = \frac{1}{\omega} \frac{(g_+(v_b) - g_-(v_b)) g_-(v_f) g_+(v_f)}{g_+(v_f) g_-(v_b) - g_+(v_b) g_-(v_f)}, \tag{4.22}
\]

with \( v_b \) as in (4.18).
Suppose first that (A5) holds. Then, using also the assumptions on \( f, g \), we have that the function \( H \in C^1 \) for any \( v_b \in (v^*, \bar{v}^*) \), as long as \( v_f \) in (4.18) belongs to \((v_{\min}, v^*)\). Indeed, the denominator does not vanish due to (3.4). The assumptions on the signs of \( g \) in (A5), as well as the fact that (4.18) implies \( v_f < v_b \), yield:

\[
g_+(v_f) g_-(v_b) - g_+(v_b) g_-(v_f) > (g_+(v_f) - g_+(v_b)) g_-(v_f) > 0.
\]

Combining this last inequality with (A5) we obtain that \( H(v_f) < 0 \).

Suppose otherwise that (B5) hold. Then we have again \( v_f < v_b \), but in this case,

\[
g_+(v_f) g_-(v_b) - g_+(v_b) g_-(v_f) < (g_+(v_f) - g_+(v_b)) g_-(v_f) < 0,
\]

so \( H(v_f) > 0 \).

Equation (4.21) combined with (4.16), (4.17) is a first order ODE for \( v_f \) for any choice of signs in (4.16), (4.17). We need to obtain a solution of equations (4.16), (4.17), (4.18) such that \( v_f(r) \in (v_{\min}, v^*) \) and \( v_b(r) \in (v^*, \bar{v}_a) \), since we need to ensure that \( c(v_f) < 0 \) and \( c(v_b) > 0 \) for all \( r > R \). However, not all these choices of sign in (4.16), (4.17) yield this type of solutions. Notice that neither \( v_f(r) \) or \( v_b(r) \) cannot reach the value \( v^* \) at a finite value of \( r > R \) since this would cause the right-hand side in equations (4.16), (4.17) to become unbounded contradicting \( \phi_f \in C^1 \). We consider separately the different possibilities.

We begin assuming that (A1)-(A6) hold. Suppose first that we make the choice of signs:

\[
\phi_b' = -1 \frac{1}{r} \sqrt{\left( \frac{r \omega}{c(v_b)} \right)^2 - 1}, \quad \phi_f' = -1 \frac{1}{r} \sqrt{\left( \frac{r \omega}{c(v_f)} \right)^2 - 1}.
\]

In this case, since \( H(v_f) < 0 \), (4.21) implies that \( v_f(\cdot) \) is increasing. We now remark that the square roots in (4.16), (4.17) cannot vanish for any \( r = R^* > R \). In the case of the choice of signs in (4.24), since \( v_f(\cdot) \) is increasing and \( v_f(r) < v^* \), it follows from (A4) that \( (c(v_f))^2 \) is decreasing, henceforth the right-hand side of (4.17) would never vanish. On the other hand, in principle, the right-hand side of (4.16) could vanish at a finite \( r \), since \( (c(v_b))^2 \) is increasing. However, locally near the value of \( r = R^* \), (4.19) would have the form:

\[
v_f' = -a_1 - a_2 \sqrt{W - v_f},
\]

for some \( a_1 > 0 \), \( a_2 > 0 \), \( W \in (v_{\min}, v^*) \). However, such a solution could not be defined smoothly for \( r > R^* \). Then, for smooth solutions, the square roots in (4.24) remain always positive for \( r > R \) and (4.21) would imply:

\[
v_f' > b, \quad R < r < \infty,
\]

for some \( b > 0 \), but this yields a contradiction, since \( v_f \in (v_{\min}, v^*) \).

In the case of the choice of signs:

\[
\phi_b' = -1 \frac{1}{r} \sqrt{\left( \frac{r \omega}{c(v_b)} \right)^2 - 1}, \quad \phi_f' = 1 \frac{1}{r} \sqrt{\left( \frac{r \omega}{c(v_f)} \right)^2 - 1},
\]

a similar argument switching the roles of \( v_f \) and \( v_b \) would also give a contradiction.

Suppose now that both signs in (4.16), (4.17) are the same. Since \( v_f(\cdot) \in C^1 \), the same argument as in the previous cases shows that \( v_f < v^* < v_b \). Notice that in these cases the right-hand side of (4.21) vanishes provided:

\[
c(v_b) + c(v_f) = 0.
\]

(4.25)
Since $c(\cdot)$ and $G(\cdot)$ are monotonically increasing, it follows from (4.18) and (4.25) that there exists a unique pair $(\bar{v}_f, \bar{v}_b)$ satisfying $v_m < \bar{v}_f < v^* < \bar{v}_b < v_s$ for which both relations (4.18) and (4.25) hold. (The uniqueness can be seen more easily using $c_b = c(v_b)$ and $c_f = c(v_f)$ as variables instead of $v_b$, $v_f$).

We now claim that neither of the square roots in the right-hand side of (4.16), (4.17) can vanish at a finite value of $r = R^* < \infty$. Indeed, suppose that only one of the roots vanishes at $r = R^*$, then, locally equation (4.21) would be:

$$v'_f = a_1 + a_2 \sqrt{\pm (v_f - W)},$$

for some $W \in (v_{min}, v^*)$, and $a_1, a_2 \in \mathbb{R}$, $a_1 \neq 0$, $a_2 \neq 0$. It can be readily checked that in such a case the function $v_f(\cdot)$ could not be extended beyond $r = R^*$. Let us assume then that both square roots in (4.16), (4.17) vanish simultaneously at $r = R^*$. The uniqueness of the pair $(\bar{v}_f, \bar{v}_b)$ then implies that $v_f(r) = \bar{v}_f$, $v_b(r) = \bar{v}_b$. Moreover, due to (4.25), $R^* \omega = c(\bar{v}_b) = -c(\bar{v}_f)$. This case can be excluded also by means of local analysis. Indeed, keeping the leading terms near $r = R^*$ we would obtain, using also (4.20), the following approximation for (4.21):

$$v'_f \approx b \left[ \sqrt{\frac{(r - R^*)}{R^*}} - \frac{c'(\bar{v}_f)}{c(\bar{v}_f)} (v_f - \bar{v}_f) \right] - \sqrt{\frac{(r - R^*)}{R^*}} - \frac{c'(\bar{v}_b)}{c(\bar{v}_b)} (v_b - \bar{v}_b)$$

for some $b \in \mathbb{R}$, $b \neq 0$. Notice that this approximation requires:

$$\frac{(r - R^*)}{R^*} - \frac{c'(\bar{v}_f)}{c(\bar{v}_f)} (v_f - \bar{v}_f) > 0,$$

$$\frac{(r - R^*)}{R^*} - \frac{c'(\bar{v}_b)}{c(\bar{v}_b)} (v_b - \bar{v}_b) > 0,$$

for $r < R^*$, but using the fact that $c(\bar{v}_f) < 0 < c(\bar{v}_b)$, as well as (A4) and (4.20), we obtain a contradiction.

We can then assume that for the solutions of (4.21) the square roots in (4.16), (4.17) are positive for $R \leq r < \infty$. Since (4.21) is a first order equation and the right hand side vanishes only for $v_f = \bar{v}_f$, the solution $v_f(\cdot)$ can be globally defined for $R \leq r < \infty$ only if $v_f(r) \to \bar{v}_f$ as $r \to \infty$, because otherwise $v_f(r)$ would reach the extremes of the interval $(v_{min}, v_s)$ for a finite value of $r$. Therefore the point $v_f = \bar{v}_f$ must be stable and this implies that the choice of signs in (4.16), (4.17) must be:

$$\phi'_b = \frac{1}{r} \sqrt{\left( \frac{r \omega}{c(v_b)} \right)^2 - 1}, \quad \phi'_f = \frac{1}{r} \sqrt{\left( \frac{r \omega}{c(v_f)} \right)^2 - 1}, \quad (4.26)$$

since in this case, linearization of (4.21) near $v_f = \bar{v}_f$ yields the approximation:

$$v'_f = -a(v_f - \bar{v}_f), \quad (4.27)$$

with $a > 0$ (in general depending on $\omega$), but whose precise formula is not very illuminating.

We now proceed to prove the existence of solutions as stated in the Theorem. Notice that, with the choice of signs (4.26), equation (4.21) becomes:

$$v'_f = \frac{H(v_f)}{r} \left( \sqrt{\left( \frac{r \omega}{c(v_f)} \right)^2 - 1} - \sqrt{\left( \frac{r \omega}{c(v_b)} \right)^2 - 1} \right) = Q(v_f, r), \quad (4.28)$$

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with $v_b$ given by (4.18).

The function $Q(v_f, r)$ has the property that:

$$Q(\bar{v}_f, r) = 0, \quad \text{for any } r \geq R.$$  \hfill (4.29)

Notice that the definition of $\bar{v}_f$ (cf. (4.18), (4.25)) implies $\bar{v}_b > \bar{v}_f$. On the other hand, since $c(\cdot)$ is monotonically increasing and $c(\bar{v}_b)c(\bar{v}_f) < 0$ due to (4.25), it follows that:

$$c(\bar{v}_b) > |c(\bar{v}_f)|.$$  

Suppose first $v_{b,0} < v^* < v_{f,0}$. Then $c(v_{b,0}), c(v_{f,0}) \neq 0$ and classical ODE theory shows the existence for $r \in [R, R + \delta]$ of a unique solution of (4.28) such that $v_f(R) = v_{f,0}$ if $\delta > 0$ is sufficiently small. The solutions of (4.28) are defined as long as both square roots remain positive and $c(v_b), c(v_f) \neq 0$. Notice that we cannot have $c(v_f(R^*)) = 0$ for any $R^* \in (R, \infty)$. Indeed, by assumption $v_f(R) < v^*$ Notice that, as mentioned before, we cannot have neither $v_f(R^*) = v^*$ or $v_b(R^*) = v^*$ because this would give singular terms in (4.16), (4.17).

We now can show that the square roots remain positive for any $r \in [R, \infty)$ for $\omega \geq \omega_0(R)$ sufficiently large. The function $Q(\cdot, r)$ is monotonically decreasing for $v_f \leq v^*$. Therefore (4.29) implies that $Q(v_f, r) < 0$ for $v_f \in (\bar{v}_f, v^*)$ and $Q(v_f, r) > 0$ for $v_f < \bar{v}_f$. It then follows that, as long as $v_f(\cdot)$ is well defined:

$$\min \{v_{f,0}, \bar{v}_f\} \leq v_f(r) < v^*, \quad r \in [R, \infty),$$

$$v^* < v_b(r) \leq \max \{v_{b,0}, \bar{v}_b\}, \quad r \in [R, \infty).$$  \hfill (4.30) \hfill (4.31)

These inequalities imply that:

$$|c(v_f)| \leq R\omega \leq r\omega, \quad r \in [R, \infty)$$

$$c(v_b) \leq R\omega \leq r\omega, \quad r \in [R, \infty)$$

if $\omega_0(R)$ is sufficiently large.

If $c(v_{f,0}), c(v_{b,0}) = 0$ one the square roots in (4.28) would diverge as $r \to R^+$. In this case the local existence of solutions of (4.28) cannot be obtained using standard ODE theory. However, the existence of a local solution can be proved by means of local analysis. Suppose by definiteness that $v_{f,0} = v^*$, since the other case is similar. Then (4.28) can be locally approximated as:

$$v_f' = \frac{H(v^*)\omega}{c'(v^*)(v^* - v_f)},$$

for $r \to R, v_f \to v^*$. Henceforth, the solutions of this problem can be approximated as:

$$(v_f - v^*) = -\sqrt{-\frac{2H(v^*)\omega}{c'(v^*)}(r - R)},$$  \hfill (4.32)

as $r \to R^+$. The local existence of this solution can be proved by means of a standard fixed point argument. Notice that since $v_f < v^*$ we must take the negative root among the two possibilities in (4.32). If $v_{b,0} = v^*$ a similar argument, using also (4.20), yields:

$$(v_b - v^*) = \sqrt{-\frac{2AH(v^*)\omega}{c'(v^*)}(r - R)} \quad \text{as } r \to R^+,$$
where:

\[ A = \frac{1}{g_+(v_{f,0})} - \frac{1}{g_-(v_{f,0})} - \frac{1}{g_+(v_f^*)} + \frac{1}{g_-(v_*)}. \]

Uniqueness of solutions with the properties stated in the theorem can also be proved in a standard way in the case \( c(v_{f,0}), c(v_{b,0}) = 0 \).

It remains to show that \( \phi_b(r) < \phi_f(r) \) for \( r \in [R, \infty) \). To this end, we use (cf. (4.21)):

\[ (\phi_f - \phi_b)' = \frac{v_f'}{H(v_f)}. \]

Using (A6) we obtain that, for \( \omega_0 \) sufficiently large:

\[ 0 < \frac{C_1}{\omega(v_b - v_f)} \leq |H(v_f)| \leq \frac{C_2}{\omega(v_b - v_f)}. \]

Using then (4.30), (4.31) it follows that:

\[ 0 < C_3 \leq |H(v_f)| \leq C_4. \]

Using then that \( v_f' \) is monotonic and \( H(v_f) < 0 \):

\[ (\phi_f - \phi_b)(r) - (\phi_f - \phi_b)(R) \geq -C_5 |v_f(r) - v_f(R)| \]

for some \( C_5 > 0 \) independent on \( \omega \). On the other hand, since \( \phi_f(R) = \omega [G_+(v_{b,0}) - G_+(v_{f,0})] \), \( \phi_b(R) = 0 \), we obtain, using that \( |v_{f,0} - v^*| + |v_{b,0} - v^*| \geq C_6 / \omega \) that:

\[ (\phi_f - \phi_b)(R) \geq C_7 > 0, \]

for some \( C_7 \) independent of \( \omega \). Since (cf. (4.30), (4.31)):

\[ |v_f(r) - v_f(R)| \leq \frac{C_8}{\omega}, \]

it then follows that:

\[ (\phi_f - \phi_b)(r) \geq C_9 > 0, \quad r \in [R, \infty). \]

If conditions (A1)-(A3), (B4)-(B7) hold, the sign of \( H(v_f) \) is now positive (cf. (4.23)) and then some of the monotonicity properties of \( v_f(\cdot), v_b(\cdot) \) change, but the same argument can be carried out along similar lines with slight modifications.

This concludes the Proof of the Theorem. \( \blacksquare \)

**Remark 2** Stability considerations that will be described in Subsection §5.1 suggest that the point where the front wave touches the stall core (point \( x_T \) in figure 1), must be a point with vanishing velocity. At such points we would have \( \lim_{r \to R} \phi_f' = \pm \infty. \)
4.3 A particular example

As an example of a system satisfying conditions (A1)-(A6) we consider the FN model described by the following functions:

\[ f(u, v) = -u(u - h(v))(u - 1), \]
\[ g(u, v) = -(v - \sigma)^2 + \Gamma uv + C, \]

where \( h(v) \) may be chosen as a linear function, \( h(v) = av + b \). We first check that this type of system satisfies all conditions for suitable values of the parameters. Indeed, conditions (A1)-(A2) are satisfied by this choice of \( f(u, v) \) by just requiring that \( h(v) \) lies between zero and one \( v \in (v_{\min}, v_{\max}) \). Then, being \( h(v) \) linear, there exists a unique point \( v^* \) satisfying condition (A3). This point is actually the value of \( v \) such that \( h(v) = 1/2 \). In this particular case, the velocity has an explicit expression that is given by

\[ c(v) = h(v) - 1/2, \]

so it is clearly monotonically increasing provided \( h(v) \) is also increasing. It then satisfies also condition (A4). As for condition (A5), we note that, in this case \( U_\pm = 0, U_1 = 1 \) and therefore,

\[ g(U_\pm(v), v) = -(v - \sigma)^2 + C, \quad g(U_1(v), v) = -(v - \sigma)^2 + \Gamma v + C, \]

so \( \bar{v}_s = \sigma - \sqrt{C} \) and \( v_s = \sigma + \sqrt{C} \), so \( C \) must be strictly positive and both values must lie between \( v_{\min} \) and \( v_{\max} \). It is then clear that \( g(U_\pm(v), v) \) satisfy the required conditions in (A5) and therefore (A6) holds as well.

Conditions (4.10)-(4.11) yield the following system,

\[ \frac{v_b - v_1}{v_b - v_2} \cdot \frac{v_f - v_2}{v_f - v_1} = \exp \left( \frac{(v_2 - v_1) \Theta}{\omega} \right), \quad (4.33) \]
\[ \frac{v_b - \sigma - \sqrt{C}}{v_b - \sigma + \sqrt{C}} \cdot \frac{v_f - \sigma + \sqrt{C}}{v_f - \sigma - \sqrt{C}} = \exp \left( \frac{2\sqrt{C} \Theta - 2\pi}{\omega} \right), \quad (4.34) \]

where \( v_1 \) and \( v_2 \) are the two roots of \( g(U_1(v), v) = 0 \). Therefore, the equations to solve are given by

\[ \phi'_b = \frac{1}{r} \sqrt{\left( \frac{r \omega}{c(v_b)} \right)^2 - 1} \quad , \quad \phi'_f = \frac{1}{r} \sqrt{\left( \frac{r \omega}{c(v_f)} \right)^2 - 1}, \quad (4.35) \]

with

\[ c(v_b) = av_b + b - 1/2 \quad , \quad c(v_f) = av_f + b - 1/2. \quad (4.36) \]

In the particular case that \( C = 1/4, \sigma = \Gamma = 1 \) and \( h(v) = 7/18 + 1/3 v \), one has that \( \bar{v}_s = 1/2, v_s = 3/2, v^* = 1/3 \), and conditions (A1)-(A6) hold for all \( v \in (3/10, 11/6) \).

As for conditions (A1)-(A3), (B4)-(B7), they are satisfied, for instance, by the following functions:

\[ f(u, v) = -u(u - h(v))(u - 1), \]
\[ g(u, v) = u - \gamma (v - v_s). \]

(A1) is satisfied for any value of \( v \in (0, 1) \) and \( U_\pm(v) = 0 < U_0(v) = X(v) < U_1(v) = 1 \) provided \( 0 < X(v) < 1 \) for \( v \in (0, 1) \) and \( X(1) = 1 \). (A2) is also satisfied. (A3) is satisfied by any point \( v^* \) such that \( h(v^*) = 1/2 \). The choice of function \( h(v) \) must then be such that it
achieves the value of 1/2 at some point \( v^* \in (0, 1) \). As before, with this particular choice of \( f(u, v) \) one can compute explicitly the velocity given by expression (3.3), and it is given by

\[
e(v) = h(v) - 1/2,
\]

so it has the same monotonicity properties as \( X(v) \). Thus, in order for (B4) to be satisfied, one must choose \( h(v) \) such that it achieves the value of 1/2 at a couple of points inside \((v_{\text{min}}, v_{\text{max}})\), and such that it decreases at the first of the velocity and it becomes increasing at the second one. For instance we may choose \( h(v) \) to be parabolic,

\[
X(v) = 1 + av(v - 1),
\]

where \( a > 2 \) to ensure that there exists two values \( v^* \) and \( v^{**} \) such that \( h(v^*) = h(v^{**}) = 1/2 \). In particular,

\[
v^* = \frac{1}{2} \left( 1 + \sqrt{1 + \frac{2}{a}} \right), \quad v^{**} = \frac{1}{2} \left( 1 - \sqrt{1 + \frac{2}{a}} \right).
\]

Condition (B5) controls the sign of \( g(U_+(v), v) \) which are in this case given by

\[
g(U_+(v), v) = 1 - \gamma(v - v_s),
\]

\[
g(U_-(v), v) = -\gamma(v - v_s).
\]

It is readily seen that \( g(U_+(v), v) > 0 \) if \( \gamma \) is small enough, and \( g(U_-(v), v) < 0 \) for any value of \( v > v_s \). It also holds in this case,

\[
g(U_-(v_1), v_1) < g(U_-(v_2), v_2), \quad g(U_+(v_1), v_1) < g(U_+(v_2), v_2),
\]

so (B6) is also satisfied.

As for condition (B7), it is readily seen that it is satisfied provided \( a < 4 \), which gives the restriction \( 2 < a < 4 \).

This case allows to compute explicitly conditions (4.10)-(4.11) which yield,

\[
\frac{1 - \gamma(v_b - v_s)}{1 - \gamma(v_f - v_s)} = e^{-\gamma\Theta}, \tag{4.37}
\]

\[
\log \left( \frac{1 - \gamma(v_b - v_s)}{1 - \gamma(v_f - v_s)} \right) - \log \left( \frac{v_b - v_s}{v_f - v_s} \right) = \frac{2\pi\gamma}{\omega}, \tag{4.38}
\]

where \( \Theta = \phi_f - \phi_b \). Upon solving (4.37)-(4.38) for \( v_b \) and \( v_f \) one finds,

\[
v_b = v_s + \frac{e^{\gamma\Theta} - 1}{\gamma(1 - e^{2\gamma})}, \tag{4.39}
\]

\[
v_f = v_s - \frac{e^{2\gamma}(1 - e^{\gamma\Theta})}{\gamma(1 - e^{2\gamma})}, \quad \text{or equivalently,} \quad v_f = v_s - e^{\gamma(2e^{-\Theta})/\omega}(v_b - v_s). \tag{4.40}
\]

Therefore, the equations to solve are simply given by

\[
\phi_b = \frac{1}{r} \sqrt{\left( \frac{r\omega}{c(v_b)} \right)^2 - 1}, \quad \phi_f = \frac{1}{r} \sqrt{\left( \frac{r\omega}{c(v_f)} \right)^2 - 1}, \tag{4.41}
\]

with

\[
c(v_b) = a(v_b(v_b - 1) + 1/2), \quad c(v_f) = a(v_f(v_f - 1) + 1/2, \tag{4.42}
\]

where \( v_b \) and \( v_f \) are given by (4.39)-(4.40). This system can be easily computed by numerical integration (see figure 3).
Figure 3: Solution of system (4.41), numerically computed with the following values for the parameters: $R = 0.2$, $\gamma = 1$, $\omega = 5$, $a = 3$, $v_s = 0.9$. As initial data we took $v_b(R) = v^* = 0.7897$, $\phi_b(R) = 0$ and $v_f(R)$ and $\phi_f(R)$ are computed with (4.39)-(4.40).

5 Analysis of the phase transition layers connecting $\Omega_s$ with $\Omega_+ \cup \Omega_-$.

A key assumption that we will make to choose the admissible fronts is that the velocity must vanish at the point of the back front intersecting with $\Sigma_s$.

5.1 Assumption $c(v) = 0$ at $x = x_N$.

Theorem 1 shows that there exist solutions of the rotating fronts satisfying the equations (4.3), (4.4), (4.10) and (4.11) for a large class of values of $R$, $\omega$, $\Theta_0$. Moreover, we can construct, at least locally, waves satisfying either the condition $c(v(x_T)) = 0$, $c(v(x_N)) = 0$ or none of them.

However, there are strong heuristic reasons to suspect that the only waves of this family that can arise as limits of solutions of the FN system (1.1) are the ones with vanishing front velocity either at $x_T$ or $x_N$. The reasons are the following:

First, kinematic constraints impose some restrictions on the sign of $c(v(x_N))$. Indeed, suppose that $c(v(x_N)) < 0$. Then, at the point $x_N$ the component of the velocity along the tangent vector to $\sigma_s$ (in the positive sense) would be negative, but this contradicts the fact that the spiral fronts are rotating at constant angular velocity $\omega > 0$. Therefore $c(v(x_N)) \geq 0$.

Suppose now that $c(v(x_N)) > 0$. A stability argument suggests that this is not possible. Indeed, at the point $x_T$ we have $c(v(x_T)) \leq 0$ and since $v$ increases in the region $\Omega_+$ it follows from Assumption (A4) that there exist a point $x_P \in \sigma_s$ such that $c(v(x_P)) = 0$. Let us denote as $I_{(x_N, x_P)}$ the subset of $\sigma_s$ that does not contain $x_T$. The monotonicity of $v$ and Assumption (A4) implies that in such an interval $c(v(x)) > 0$.

Let us suppose that there exist solutions of (1.1) converging at traveling waves as in Section 4. At the points close to $I_{(x_N, x_P)}$ there would be boundary layers connecting the states $(u, v) =$
(\(u_s, v_s\)) with the states \((u, v) = (U_+(v), v)\). We claim, that such connections should be unstable for a large class of nonlinearities \(f, g\). The reason, is that we can expect that small perturbations of the steady solution connecting \((u_s, v_s)\) with \((U_+(v), v)\) would split in a steady solution connecting the states \((u_s, v_s)\) and \((U_-(v), v)\) followed by a traveling wave connecting \((U_-(v), v)\) with \((U_+(v), v)\) because this would be a more favorable from the energetic point of view. Since \(c(v) > 0\) the region where \((u, v) = (U_-(v), v)\) would then to expand. A similar conclusion can be achieved by means of energy arguments. Therefore the set \(I_{(x_N, x_P)}\) would naturally disintegrate. It would be then natural to assume that \(c(v(x_N)) = 0\).

A natural question to ask is why a similar argument cannot be applied along the interface \(\sigma_f^-\). Indeed, along such interface the boundary layer connecting the regions \(\Omega_+\) and \(\Omega_-\) would be a steady solution connecting \((u_s, v_s)\) with \((U_-(v), v)\). Notice, however, that such a solution, in general, is not an absolute minimizer for the energy \(F(U_-(v), v) > F(U_+(v), v))\). It is reasonable to ask why a connection of the minimizer of the energy \(u_s\) with the metastable state \(U_+(v)\) considered in the argument above, tends to break in a steady state plus a wave, and this is not the case in the case of the connection between the minimizer \(u_s\) and the metastable state \(U_-(v)\). It seems plausible that this will be due to the fact that in order to produce the breakdown of the steady state in this second case, large perturbations of \(U\) in some regions would be needed. In other words, the steady solution connecting \((u_s, v_s)\) with \((U_+(v), v)\) would be unstable for a large class of nonlinearities \(f, g\) and the steady solution connecting \((u_s, v_s)\) with \((U_-(v), v)\) would be metastable.

The core region as it is sketched in figure 1 is composed of different transition layers whose existence and stability must be studied.

In the next two subsections we rigorously construct the layers connecting the stall region of the core with the outer quiescent and excited regions.

### 5.2 On the existence of the boundary layers connecting different values of \(v\). Connections \(u_s\) to \(U_+ (v)\)

We consider a point \(x_P\) placed in one of the interfaces where \(v\) is discontinuous in the limit \(\varepsilon \to 0\). We assume that the curves \(\sigma_s, \sigma_b, \sigma_f\) are smooth, and therefore normal and tangent vectors at \(x_P\) can be defined. We can assume that the coordinates \(x_1, x_2\) are oriented respectively in the directions of the normal and tangent vectors at \(x_P\) and also that \(x_P = 0\). We then define a new stretched coordinate by means of:

\[
x_1 = \varepsilon \xi,
\]

as well as a natural time coordinate:

\[
t = \varepsilon \tau.
\]

Then, keeping just the leading order terms in (1.1):

\[
\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \xi^2} + f(u, v), \quad \frac{\partial v}{\partial \tau} = 0.
\] (5.1)

In this time scale \(v = v(\xi, \tau)\) remains frozen and \(u\) must approach to equilibrium in times \(\tau\) of order one. Therefore, after some transient stage, the form of \(u\) in this region would be given by:

\[
\frac{\partial^2 u}{\partial \xi^2} + f(u, v(\xi)) = 0.
\] (5.2)

We will check later that it is self-consistent to have variations of \(v\) over distances of order one in the coordinate \(\xi\). Our goal is to obtain a function \(u = u[v](\xi)\), solution of (5.2). In the
Suppose also that
\[ u \rightarrow U_\pm (v_{\pm\infty}) \text{ as } \xi \rightarrow -\infty \text{ with } c(v_{-\infty}) < 0, \]  
\[ u \rightarrow U_\pm (v_{\infty}) \text{ as } \xi \rightarrow \infty \text{ with } c(v_{\infty}) > 0. \]  
(5.3) (5.4)

Intuitively the potential function \( F(\cdot, v) \) has a global minimum for \( v = v_{-\infty} \) and \( v = v_{\infty} \). The minimizer of the potential energy can be expected to be a function \( u(\cdot) \) connecting these global minima. The following Theorem makes this statement precise.

**Theorem 3** Suppose that \( v = v(\xi) \) is differentiable and it satisfies:
\[ \int_{-\infty}^{\infty} |v_\xi|^2 \, d\xi < \infty, \quad \lim_{\xi \rightarrow -\infty} v(\xi) = v_{-\infty}, \quad \lim_{\xi \rightarrow \infty} v(\xi) = v_{\infty}, \]  
with
\[ c(v_{-\infty}) > 0, \quad c(v_{\infty}) < 0. \]  
(5.5) (5.6)

Suppose also that
\[ f'(U_-(v_{-\infty})) < 0, \quad f'(U_+(v_{+\infty})) < 0. \]  
(5.7)

Then, there exists a solution \( u = u[v(\xi)] \) of (5.2) satisfying (5.3), (5.4). Moreover, \( u = \bar{u}(\xi) + \varphi(\xi) \) where \( \varphi \in H^1(\mathbb{R}) \) is a minimizer of the functional:
\[ I[\varphi] = \int_{-\infty}^{\infty} \left[ \frac{1}{2} (\varphi_\xi)^2 + \bar{\varphi}_\xi \varphi_\xi + (F(\bar{u} + \varphi, v) - F(\bar{u}, v)) \right] \, d\xi, \]

and \( \bar{u} \) is a \( C^1 \) function such that:
\[ \bar{u}(\xi) = U_-(v(\xi)) \quad \text{if} \quad |v(\xi) - v_{-\infty}| < \frac{v^* - v_{-\infty}}{5}, \]  
\[ \bar{u}(\xi) = U_+(v(\xi)) \quad \text{if} \quad |v(\xi) - v_{\infty}| < \frac{v_{\infty} - v^*}{5}. \]  
(5.8) (5.9)

**Proof.** Formula (5.6) as well as (5.7) give:
\[ F(\bar{u} + \varphi, v) - F(\bar{u}, v) \geq \theta \varphi^2 - \psi(\xi), \]  
(5.10)

where \( \theta > 0 \) and due to the limit formulas in (5.5), \( \psi(\cdot) \) is supported in the compact set where \( |v(\xi) - v_{-\infty}| \geq \frac{v^* - v_{-\infty}}{5} \) and \( |v(\xi) - v_{\infty}| \geq \frac{v_{\infty} - v^*}{5} \).

Notice that (5.8), (5.9) as well as (5.5) imply \( \int_{-\infty}^{\infty} |\bar{\varphi}_\xi|^2 \, d\xi < \infty \). Therefore, the functional \( I[\varphi] \) can be estimated as:
\[ I[\varphi] \geq \frac{1}{2} \int_{-\infty}^{\infty} (\varphi_\xi)^2 - \frac{1}{4} \int_{-\infty}^{\infty} (\varphi_\xi)^2 - \int_{-\infty}^{\infty} |\bar{\varphi}_\xi|^2 + \theta \int_{-\infty}^{\infty} \varphi^2 - \int_{-\infty}^{\infty} \psi(\xi). \]

Then \( I[\varphi] \geq a \|\varphi\|_{H^1(\mathbb{R})}^2 - b \) for some \( a > 0, b \in \mathbb{R} \). Since the functional \( I[\cdot] \) is quadratic in the derivative \( \varphi_\xi \) it follows, from classical calculus of variations results, the existence of a minimizer of \( I[\cdot] \). Let us denote as \( \bar{\varphi} \) such a minimizer. It can be readily checked that \( u = \bar{u} + \bar{\varphi} \) is a weak solution of (5.2). Classical regularity theory for elliptic equations shows that \( u \) is a classical solution of (5.2). Moreover we also have \( \bar{\varphi}(\xi) \rightarrow 0 \) as \( |\xi| \rightarrow \infty \), whence (5.3), (5.4) follow. \( \blacksquare \)

We also give an alternative proof with a dynamical systems flavor, since it provides some interesting insight in the structure of these layers.
This proof requires more stringent assumptions on the decay of $v(\xi)$ as $|\xi| \to \infty$. We will assume that there exists a $\delta > 0$ and a $K > 0$ such that

$$|v(\xi) - v_{-\infty}| < Ke^{(\lambda_- + \delta)\xi}, \quad \xi \leq 0,$$

$$|v(\xi) - v_{\infty}| < Ke^{-(\lambda_+ + \delta)\xi}, \quad \xi \geq 0,$$

where $\lambda_- = \sqrt{-f_u(U_-(v_{-\infty}), v_{-\infty})}$, $\lambda_+ = \sqrt{-f_u(U_+(v_{\infty}), v_{\infty})}$.

**Theorem 4 (Existence of the layer solution.)** Suppose that $v(\xi)$ satisfies (5.11), (5.12) with $v_{-\infty}$, $v_{\infty}$ satisfying (5.6). Then, there exists at least one solution of (5.2) such that

$$\lim_{\xi \to -\infty} u(\xi) = U_- (v_{-\infty}), \quad \lim_{\xi \to \infty} u(\xi) = U_+ (v_{\infty}) .$$

**Proof.** We can rewrite (5.2) as:

$$\frac{\partial^2 u}{\partial \xi^2} - (\lambda_-)^2 (u - U_- (v_{-\infty})) + R_- (u, \xi) = 0,$$

where:

$$R_- (u, \xi) = [f (u, v(\xi)) - f (u, v_{-\infty})] +$$

$$+ \left[ f (u, v_{-\infty}) - f (U_-(v_{-\infty}), v_{-\infty}) + (\lambda_-)^2 (u - U_- (v_{-\infty})) \right] .$$

We are interested in finding a solution of (5.14) satisfying the first identity in (5.13). To this end, we reformulate (5.14), using variation of constants, as:

$$u(\xi) = U_- (v_{-\infty}) + C_0 e^{\lambda_- \xi} - \int_{-\infty}^{\xi} \frac{\sinh (\lambda_- (\xi - \xi'))}{\lambda_-} R_- (u(\xi'), \xi') d\xi'$$

$$\equiv T[u](\xi) ,$$

where $\xi_0$ is a fixed number to be precised that could depend on $C_0$.

Using (5.15) as well as our assumptions on $f$ we obtain:

$$|R_- (u, \xi)| \leq C \left[ (u - U_- (v_{-\infty}))^2 + |v(\xi) - v_{-\infty}| \right]$$

for $|u - U_- (v_{-\infty})| \leq \delta_0$ with $\delta_0 > 0$ sufficiently small. Therefore, due to (5.11) we can define the operator $T[u](\xi)$ by means of (5.16) in the class of functions $u$ satisfying:

$$|u - U_- (v_{-\infty})| \leq Me^{\lambda_- \xi} ,$$

for some $M > 0$. A standard fixed point argument shows the existence of a unique solution of (5.16) in the class of functions (5.17) for $\xi \leq \xi_0$ with $\xi_0$ depending on $C_0$. We will denote this solution from now on as $u(\cdot; C_0)$.

Hypothesis (5.6) implies that the phase portraits associated to the equations

$$\frac{\partial^2 u}{\partial \xi^2} + f(u, v_{-\infty}) = 0,$$

$$\frac{\partial^2 u}{\partial \xi^2} + f(u, v_{\infty}) = 0,$$

are as the ones on the left and on the right in Figure 4 respectively. Notice that if $|C_0|$ is sufficiently large, the trajectories associated to (5.2) become close, as long as $u, u_\xi$ remain
bounded, to those of (5.18). In particular, modifying the shape of \( f(u,v) \) for large values of \( u \), we can assume that \( f(u,v) \) behaves like \(-u^3\) for \( |u| \to \infty \). Therefore, if \( C_0 < 0 \), and \(|C_0| \) is large we obtain that \( u(\xi; C_0) \to -\infty \) as \( \xi \to \xi_1 < \infty \). Similarly \( u(\xi; C_0) \to \infty \) as \( \xi \to \xi_2 < \infty \) if \( C_0 > 0 \) with \(|C_0| \) large.

We now define the following set:

\[
\mathcal{C}_+ = \{ C_0 \in \mathbb{R} \text{ such that } u(\xi; C_0) \to +\infty \text{ as } \xi \to \xi^* = \xi^*(C_0) < \infty, \forall C_0 \geq C_0 \}.
\]

The sets \( \mathcal{C}_+ \) and \( \mathbb{R} \setminus \mathcal{C}_+ \) are not empty due to the previous argument. Therefore due to the definition of \( \mathcal{C}_+ \) we have that \( \mathcal{C}_+ \) is bounded from below. Let us write:

\[
C_0^* = \inf \mathcal{C}_+.
\]

Then \(-\infty < C_0^* < \infty \). We now claim that the trajectory \( u(\xi; C_0^*) \) satisfies:

\[
\lim_{\xi \to +\infty} u(\xi; C_0^*) = U_-(v_\infty), \quad \lim_{\xi \to -\infty} u(\xi; C_0^*) = U_+(v_\infty).
\]  

(5.20)

Indeed, the first identity in (5.20) follows from the definition of \( u(\xi; C_0^*) \). On the other hand, the second identity in (5.20) can be obtained as follows. If \( \lim_{\xi \to -\infty} u(\xi; C_0^*) = -\infty \), a continuity argument would show that \( \lim_{\xi \to -\infty} u(\xi; C_0^* + \delta) = -\infty \) for some \( \delta > 0 \), contradicting the definition of \( C_0^* \). If \( \lim_{\xi \to +\infty} u(\xi; C_0^*) = +\infty \), continuity would yield also \( \lim_{\xi \to -\infty} u(\xi; C_0) = +\infty \) for any \( C_0 \in (C_0^* - \varepsilon_0, C_0^*) \) for some \( \varepsilon_0 > 0 \), and this would contradict again the fact that \( C_0^* \) is the infimum of \( \mathcal{C}_+ \). The phase portrait on the right-hand side of Figure 4 shows then that one of the following three possibilities takes place:

\[
\lim_{\xi \to +\infty} u(\xi; C_0^*) = U_-(v_\infty), \qquad (5.21)
\]

\[
\lim_{\xi \to -\infty} u(\xi; C_0^*) = U_+(v_\infty), \qquad (5.22)
\]

\[
\lim_{\xi \to +\infty} E(u(\xi; C_0^*) \cdot u_\xi(\xi; C_0^*)) < -F(U_-(v_\infty), v_\infty), \qquad (5.23)
\]

where:

\[
E(u, p) = \frac{p^2}{2} - F(u, v_\infty).
\]

Notice that in the cases (5.21), (5.23) it would be possible to find some \( C_0 > C_0^* \) such that for large values of \( \xi \), \( E(u(\xi; C_0^*) \cdot u(\xi; C_0^*)) \in (-F(U_-(v_\infty), v_\infty), -F(U_+(v_\infty), v_\infty)) \).

For such trajectories an examination of the phase portrait in the right of Figure 4 shows that \( u(\xi; C_0) = -\infty \), contradicting the definition of \( C_0^* \) (see definition of \( \mathcal{C}_+ \) above). Then (5.22) holds and the Theorem follows.

5.3 On the existence of the boundary layers connecting different values of \( v \).

Connections \( u_s \) to \( U_-(v) \)

**Theorem 5** Suppose that \( v = v(\xi) \) is differentiable and it satisfies:

\[
\int_{-\infty}^{\infty} |v_\xi|^2 d\xi < \infty, \quad \lim_{\xi \to -\infty} v(\xi) = v_{-\infty}, \quad \lim_{\xi \to +\infty} v(\xi) = v_{+\infty}
\]  

(5.24)

with

\[
c(v_{-\infty}) > 0, \quad c(v_{+\infty}) > 0
\]  

(5.25)
Let \( \bar{u} \) be:

\[
\bar{u} (\xi) \equiv U_0(v(\xi)) - v(\xi)
\]

(5.26)

Given \( F \) as in (3.1) we define a function \( H \) by means of:

\[
H (\varphi, \xi) = \begin{cases} 
F (\bar{u} (\xi) + \varphi, v(\xi)) - F (\bar{u} (\xi), v(\xi)), & \varphi \leq U_0(v) - U_-(v) \\
G (\varphi, \xi), & \varphi > U_0(v) - U_-(v)
\end{cases}
\]

where \( G \) is chosen to make \( H \in C^1 \), as well as:

\[
\lim_{\varphi \to \infty} \left[ \inf_{\xi \in \mathbb{R}} H (\varphi, \xi) \right] = \infty, \\
H (\varphi, \xi) \geq F (U_0(v(\xi)), v(\xi)) - F (U_-(v(\xi)), v(\xi)), & \varphi > U_0(v) - U_-(v)
\]

Let us define the functional:

\[
I [\varphi] = \int_{-\infty}^{\infty} \left[ \frac{1}{2} (\varphi_\xi)^2 + \bar{u}_\xi \varphi_\xi + H (\varphi, \xi) \right] d\xi,
\]

where \( \varphi \in H^1(\mathbb{R}) \). Let \( M \) be:

\[
M = \inf \{ U_0(v(\xi)) - U_-(v(\xi)) : \xi \in \mathbb{R} \}.
\]

Suppose that:

\[
\beta \equiv \inf \left\{ I [\varphi] : \varphi \in H^1(\mathbb{R}) \text{ such that } \sup_{\xi \in \mathbb{R}} \varphi \geq M \right\} > 0.
\]

(5.27)

Then, there exists a solution \( u = u[v](\xi) \) of (5.2) satisfying:

\[
u(\xi) \to U_-(v_{-\infty}) \text{ as } \xi \to -\infty,
\]

\[
u(\xi) \to U_-(v_{+\infty}) \text{ as } \xi \to +\infty.
\]

Moreover, \( u = \bar{u}(\xi) + \varphi(\xi) \) where \( \varphi \in H^1(\mathbb{R}) \) is a minimizer of the functional \( I [\varphi] \).
Proof. - Classical arguments in Calculus of Variations show the existence of a minimizer for \( I[\varphi] \), \( \varphi \in H^1(\mathbb{R}) \) (cf. Proof of Theorem 3). Moreover, \( u = \bar{u}(\xi) + \varphi(\xi) \) satisfies the corresponding Euler-Lagrange equation. In order to show that such equation reduces to (5.2) it suffices to show that \( \varphi < M \). Suppose that there exists \( \xi_0 \in \mathbb{R} \) such that \( \varphi(\xi_0) \geq M \). Then, the definition of \( \beta \) in (5.27) shows that \( I[\varphi] > 0 \), but this contradicts the fact that \( \varphi \) is a minimizer of \( I[\varphi] \), since \( I[0] = 0 \).

Remark 6 Condition (5.27) basically means that \( \bar{u}(\xi) \) is sufficiently flat and that the energy barrier between \( U_-(v) \) and \( U_0(v) \) is large enough. Indeed, suppose that \( \frac{dU_-(v)}{dv} = 0 \). Then, \( \bar{u} \) is constant and due to (5.7),

\[
I[\varphi] \geq \int_{-\infty}^{\infty} \left[ \frac{1}{2} (\varphi')^2 + a\varphi^2 \right] d\xi, \quad a > 0.
\]

Since \( M > 0 \) it then follows that \( \beta > 0 \). A similar result can be obtained under weaker assumptions on \( U_-(v) \). Let us assume that \( H(\varphi, \xi) \geq a\varphi^2 \) for some \( a > 0 \) as well as \( \int_{-\infty}^{\infty} (\bar{u}_\xi)^2 d\xi \) sufficiently small. Combining Young’s and Sobolev’s inequalities we obtain:

\[
I[\varphi] = \int_{-\infty}^{\infty} \left[ \frac{1}{4} (\varphi')^2 + a\varphi^2 \right] d\xi - \int_{-\infty}^{\infty} (\bar{u}_\xi)^2 d\xi \geq BM - \int_{-\infty}^{\infty} (\bar{u}_\xi)^2 d\xi,
\]

and the right-hand side of this inequality is positive if \( \int_{-\infty}^{\infty} (\bar{u}_\xi)^2 d\xi \) is sufficiently small.

These are “metastable” solutions. They are obtained by means of a minimization argument after modifying the potential energy \( F(u, v) \) to a new one having only minima at \( U_-(v) \). If the variation of \( U_-(v) \) with respect to \( v \) is not too strong, the function \( u \) would change very little and therefore it would not arrive to the modifications made in \( F \). The construction of these solutions can be made using variational arguments, but also with a phase portrait.

In this case the transition takes place in the same branch of \( f(u, v) = 0 \). This should not change things much. Proceeding as before, the equation for this layer is again equation (5.2) but the boundary conditions are now given by \( u \to U_+ \) as \( \xi \to -\infty \) and \( u \to U_-(v(\xi, t_0)) \) as \( \xi \to -\infty \). The same stability criterion should apply for this layer, which means, again, that the piece of branch \( U_-(v) \) that is covered should never go beyond \( v = 0 \).

Theorem 7 (Existence of the layer solution) Suppose that \( v(\xi) \) satisfies (5.11), (5.12) with \( v_{-\infty}, v_{\infty} \) satisfying (5.6). Then, there exists at least one solution of (5.2) such that

\[
\lim_{\xi \to -\infty} u(\xi) = U_-(v_{-\infty}), \quad \lim_{\xi \to \infty} u(\xi) = U_-(v_{\infty}). \quad (5.28)
\]

Proof. The existence of solutions \( u(\cdot; C_0) \) such that \( u(\xi; C_0) \sim C_0 e^{\lambda - \xi} \) as \( \xi \to -\infty \) can be obtained exactly as in the proof of Theorem 4. We now define the set:

\[ C_\ast = \{ C_0 \in \mathbb{R} \text{ such that } u(\xi; C_0) \to -\infty \text{ as } \xi \to \xi^\ast(\bar{C}_0) < \infty, \forall \bar{C}_0 \leq C_0 \}. \]

We define \( C_0^\ast = \sup C_\ast \). Arguing by continuity, as in the Proof of Theorem 4, it follows that \( |u(\xi; C_0^\ast)| \) remains bounded as \( \xi \to \infty \). We then have the following possibilities:

\[
\lim_{\xi \to \infty} u(\xi; C_0^\ast) = U_-(v_{\infty}), \quad \lim_{\xi \to \infty} u(\xi; C_0) = U_+(v_{\infty}) \quad (5.29)
\]

\[
\lim_{\xi \to \infty} E(u(\xi; C_0^\ast), u_\xi(\xi; C_0^\ast)) < -F(U_-(v_{\infty}), v_{\infty}). \quad (5.31)
\]
In the cases (5.30), (5.31) it is possible to find $C_0 < C^*_0$ such that the corresponding trajectory $(u(\xi;C_0), u_{\xi}(\xi;C_0))$ remains trapped in the region where $E(u(\xi;C^*_0), u_{\xi}(\xi;C^*_0)) < -F(U_-(\nu_\infty), v_\infty)$. This would contradict the definition of $C_-$, $C^*_0$ whence (5.29) and the Theorem follows.

Remark 8 Using, either the variational argument or the dynamical systems approach, it is possible to find trajectories connecting the points $U_-(\nu_\infty)$ and $U_-(\nu_\infty)$ if we assume $c(\nu_\infty) > 0$ and $c(\nu_\infty) > 0$.

6 Some ideas concerning Young’s condition and the triple junction.

Near the triple junction we can use two different groups of variables, namely the original space coordinate $x$, and the rescaled coordinate $\xi = x - x_0/\varepsilon$. The time scales suggest that $v$ is frozen in the triple junction in a natural time scale $\tau = t/\varepsilon$. The problem in the ”inner coordinates” reduces to:

$$\frac{\partial u}{\partial \tau} = \Delta_\xi u + f(u, v(\xi)),$$

where $v(\xi)$ matches asymptotically for, say $\xi_2 \rightarrow \infty$ with the condition $v_f(R)$ computed for the dynamics of the spirals. To the leading order $v$ is just a function of $\xi_2$ that is the normal coordinate to the ”discontinuity line for $v$”. We recall that there are not true discontinuities for $v$, but just a variation of it over lengths $\xi$ of order one.

This equation is a gradient flow generated by the functional:

$$\int \left[ \frac{\|\nabla u\|^2}{2} + F(u, \xi) \right] dx,$$

with the potential energy:

$$F(u, \xi) = -\int_0^u f(s, v(\xi)) ds,$$

and the metric generated by the $L^2$ scalar product.

In the case of energies concentrated along curves or surfaces, it is well known that minimization of the energy yields Young’s condition (c.f. [5], [10]). This is not possible for the problem under consideration because there is a natural width for the size of the region where the energy is concentrated, that is the size of the region where $v$ has relevant changes, which is similar to the size of the region where $u$ changes abruptly. The key difficulty is that the main dynamics of these waves is driven by the energy bulk. It is interesting to remark that there is some kind of selection mechanism for the asymptotic angles for the interfaces in the case of waves associated to gradient flows generated by simpler functionals than (6.1), namely,

$$\sum_{k=1}^{3} \sigma_k H^1(\gamma_k) + \sum_{\ell=1}^{2} \alpha_k |\Omega_\ell|,$$

where $\gamma_k$ are interfaces separating the domains $\Omega_1$, $\Omega_2$ as well as these domains from a fixed boundary. The problem described by this functional is the dynamics of interfaces whose area is proportional to the energy. The coefficients $\sigma_k$, $\alpha_k$ are the interface and bulk energies respectively. The Riemannian structure is generated by the scalar product of infinitesimal variations.
along the normal. We will assume that the effect of junctions is zero, since infinitesimal variations would affect a set of zero measure under suitable transversality conditions.

In this case the Young condition holds not just for steady states, but for the whole gradient flow. The argument is the following. We can modify slightly the interface separating the phases $\Omega_1, \Omega_2$ near the triple junction. The change can be restricted to a region at a distance $\delta$ of the junction, and to affect to a curve of length $\delta$. If Young’s condition is not satisfied this would provide an increment of the energy of order $\delta$, due to the change of the three interfaces $\gamma_k$. The change of energy would be smaller only if Young’s condition holds. However, such a change would give a change of energy due to the bulk terms as well as the term coming from the scalar product (term containing the time derivative) of order $\delta^2$, because the change affects only a region of this area, and these changes are proportional to the affected area. Therefore, in order to have the cancellation of the terms of order $\delta$ we must have the Young’s condition. Or more precisely, if the Young’s condition is not satisfied, velocities of order $\frac{1}{\delta}$ would be generated in very force transient times.

However, the situation is more complicated in our case. Functional (6.1) can be approximated, in the original variables $x$, by something like:

$$\varepsilon \sum_{k=1}^{3} \sigma_k H^1(\gamma_k) + \sum_{\ell=1}^{2} \alpha_k |\Omega_\ell|.$$  

For this type of functional the Young’s condition would be satisfied, but only in a layer of order $\varepsilon$. Unfortunately, rescaling $\varepsilon$, and therefore using $\xi$ as variable, it turns out that the width of all the interfaces (two phases $\Omega_+, \Omega_-$, and also the side wall, or interface with $\Omega_s$) become of order one. In other words, there is not a true interface for the length scales where the ”surface” and ”bulk” terms are comparable. Then, the energies must be studied for the whole system.

We remark, however, that it is interesting to examine in which cases a selection of the angle takes place for gradient flows generated by the functional (6.2) takes place. The corresponding evolution of equations for the interfaces are:

$$V_n = \kappa + |c|.$$  \hfill (6.3)

We assume that the motion of the curve takes place in the half-plane $\{x_1 > 0\}$. For the gradient flow under consideration Young’s condition holds at the boundary $\{x_1 = 0\}$. If we parametrize the curves as $x_2 = f(x_1, t)$ the evolution can be written as:

$$\frac{\partial f}{\partial t} = \frac{f_{x_1 x_1}}{1 + (f_{x_1})^2} + c\sqrt{1 + (f_{x_1})^2},$$

and Young’s condition in this case just means:

$$f_{x_1}(0, t) = \tan(\theta_Y).$$

We look for traveling waves with the form:

$$f(x_1, t) = Vt + \varphi(x_1),$$

where, since $|c| > 0$ we have $V > 0$. Therefore $\varphi(x_1)$ satisfies:

$$V = \frac{\varphi_{x_1 x_1}}{1 + (\varphi_{x_1})^2} + c\sqrt{1 + (\varphi_{x_1})^2}.$$
This equation can be reduced to a first order ODE with the change of variables:

\[ z = \varphi x_1, \]
\[ z_{x_1} = (1 + z^2) \left( V - c\sqrt{1 + z^2} \right). \]

We need \( V > c \). Moreover, there are two steady states, namely:

\[ z_+ = \sqrt{\left( \frac{V}{c} \right)^2 - 1}, \quad z_- = -\sqrt{\left( \frac{V}{c} \right)^2 - 1}. \]

The steady state \( z_+ \) attracts the trajectories. If we have \( z(0) = \tan (\theta_Y) > z_- \) we have \( z(x) \to z_+ \text{ as } x \to \infty \). On the other hand, the only solution that converges to \( z_- \) as \( x \to \infty \) is \( z = z_- \). Therefore, in the case of the behavior \( z_- \) the Young’s angle \( \theta_Y \) prescribes uniquely the speed, since we have:

\[ \tan (\theta_Y) = -\sqrt{\left( \frac{V}{c} \right)^2 - 1}. \]

Since (hopefully) \( \omega \) is uniquely fixed by means of the ”eigenvalue problem” for the evolution of \( v \) on the disc, we would obtain then that \( V \) is also fixed and:

\[ V = R\omega, \]

and this would give \( R \).

The problem is that the flow generated by the functional (6.2) is not the same as the one generated by the functional (6.1) and unfortunately it is not possible to approximate (6.1) by (6.2) because the natural length scale where area and bulk terms are comparable is \( \xi \) of order one and in this scale there are not true interfaces, but regions of variation of \( u \) of width one.

We do not know if a similar selection of the asymptotic angle takes place for interfaces evolving according to the gradient flow generated by (6.1). This is a question that would be worth to settle mathematically. It is important to remark that the key difference with the case described by (6.2) is that a true Young contact angle cannot be expected, but just some kind of diffuse interface yielding, far away from the region where \( v(x) \) changes, a planar interface with a defined asymptotic angle.

7 On the dynamics of \( v \) near the discontinuity lines.

We first define the geometrical objects and functions which will be required to describe the limit model of the FN equation (1.1) in the limit \( \varepsilon \to 0 \).

(1) A set of disjoint domains \( \mathcal{F}_k (t) \), \( k = 1, ..., N \) that describes the regions of continuity of \( v \). The union of these domains is the whole space (except for the boundaries).

(2) A function \( v(x,t) \) that is continuous in each of the domains \( \mathcal{F}_k (t) \).

(3) The domains \( \mathcal{F}_k (t) \) can be partitioned in a set of domains \( U^k_+ (t) \), \( U^k_- (t) \) where \( u_\varepsilon \) tends to the limit \( U_+ (v) \), \( U_- (v) \) respectively.

(4) Interfaces \( \gamma_{i,j} = \mathcal{F}_i (t) \cap \mathcal{F}_j (t) \). They are the boundaries where \( v \) is discontinuous. It seems that these interfaces do not change for this specific rescaling in times \( t \) of order one. (This must be checked, and explore if other rescalings could have different limits).

The evolution of these sets will be given in the next Section. We now describe first how is the evolution of the functions \( v \), \( u \) near the interfaces \( \gamma_{i,j} \). To this end, we introduce a set
of spatial variables along the tangential and normal directions to $\gamma_{i,j}$. Let us denote as $x_n$ a coordinate along the normal direction with $x_n = 0$ for $x \in \gamma_{i,j}$. Given any function $\Psi$ defined in a neighborhood of the form $x_n > 0$ we can compute its limit as $x_n \to 0^+$. We will label the limit values computed in this form as limits for $(\gamma_{i,j})^+$. We can define limits in $(\gamma_{i,j})^-$ similarly, by means of limits from $x_n < 0$. The structure of the equations (1.1) suggests to stretch the normal coordinate as:

$$x_n = \varepsilon \xi$$

Then (1.1) becomes, to the leading order

$$\varepsilon \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \xi^2} + f(u, v), \quad (7.1)$$
$$\frac{\partial v}{\partial t} = g(u, v). \quad (7.2)$$

The characteristic time scale associated to (7.1) is much shorter than the one associated to (7.2). We can then assume that $u$ is at equilibrium for the time scale $t$. The functions $u, v$ are functions that in this interface are defined in $\gamma_{i,j} \times \mathbb{R} \times [0, \infty)$. Therefore we can approximate (7.1), (7.2) as:

$$\frac{\partial^2 u}{\partial \xi^2} + f(u, v) = 0, \quad \frac{\partial v}{\partial t} = g(u, v). \quad (7.3)$$

The first equation in (7.3) defines a functional $v \to u[v]$, although it must be complemented with suitable conditions for $u$ as $\xi \to \pm \infty$. These conditions depend on the type of domain $\mathcal{U}_+(t), \mathcal{U}_-(t)$ intersecting with the sides of the interfaces $(\gamma_{i,j})^+, (\gamma_{i,j})^-$. More precisely, we must assume:

$$u \to U_+(v) \text{ as } \xi \to \infty \text{ if } (\gamma_{i,j})^+ \subset \mathcal{U}_+(t),$$
$$u \to U_-(v) \text{ as } \xi \to -\infty \text{ if } (\gamma_{i,j})^- \subset \mathcal{U}_+(t),$$
$$u \to U_+(v) \text{ as } \xi \to -\infty \text{ if } (\gamma_{i,j})^- \subset \mathcal{U}_-(t).$$

This defines uniquely the functional $v \to u[v]$ except at the triple junctions. This will not pose a serious problem unless the triple junctions are at rest.

We then have that the function $v$, that in this boundary layer will be denoted as $V(x, \xi, t)$ for $(x, \xi, t) \in \gamma_{i,j} \times \mathbb{R} \times [0, \infty)$, evolves according to the equation:

$$\frac{\partial V}{\partial t} = g(u[V], V) \equiv \mathcal{G}[V; U_+(t), U_-(t), \gamma_{i,j}]. \quad (7.4)$$

In the particular case of the rigidly rotating spiral waves considered in Section 4 the functional equation (7.4) becomes:

$$-\omega \frac{\partial V}{\partial \phi} = \mathcal{G}[V; \Omega_+, \Omega_-, \partial B_R(0)]. \quad (7.5)$$

It is important to remark that in this case the choice of the asymptotic values for $u$ as $\xi \to \pm \infty$ in (7.3) takes a particularly simple form:

$$u \to u_s, \quad \xi \to -\infty,$$
$$u \to U_+(v), \quad \xi \to \infty \text{, } 0 < \phi < \phi_f(R) = \Phi(\omega),$$
$$u \to U_-(v), \quad \xi \to \infty \text{, } \Phi(\omega) = \phi_f(R) < \phi < 2\pi,$$
where by assumption $\phi_b (R) = 0$ and $\phi_f (R)$ is defined by means of (4.10), (4.11) with $r = R$. Notice that $\phi_f (R) = \Phi (\omega)$ is independent on $R$, and it depends only on $\omega$. Therefore, the problem (7.5) has the functional form:

$$-\omega \frac{\partial V}{\partial \phi} = G[V; \omega].$$  \hspace{1cm} (7.6)

Our goal is to obtain a solution of this problem with $V = V(\phi, \xi) = V(\phi + 2\pi, \xi)$. Notice that the boundary has been parametrized using just the coordinate $\phi$. It is important to remark that this problem is independent on $R$.

8 Formulation of the limit model.

We now describe the evolution of the geometrical objects and functions $\mathcal{F}_k (t), v(x,t), \mathcal{U}^k_+ (t), \mathcal{U}^k_- (t), \gamma_{i,j}$ defined in Section 7 in the limit $\varepsilon \to 0$ of the model (1.1). We need to introduce an additional function $V(x,\xi,t)$ defined in $\gamma_{i,j} \times \mathbb{R} \times [0, \infty)$. The variable $\xi$ is a stretched normal coordinate at each point $x \in \gamma_{i,j}$. The function $V(x,\xi,t)$ changes according to an evolution law:

$$V_t (x,\xi,t) = g(U[V(x,\cdot,t), V])$$

that is a "generalization" of the laws in the regions $\mathcal{U}^k_+ (t), \mathcal{U}^k_- (t)$ where $u = U_\pm$. The functional $U[V]$ gives the profile of $u$ in the interface. This depends non locally on $V$.

The evolution of the different magnitudes is then the following one:

(I) The evolution equation for $v$ in the regions $\mathcal{F}_k (t)$ is:

$$v_t = g(U_\pm(v), v).$$

(II) The interfaces between $\mathcal{U}^k_+ (t), \mathcal{U}^k_- (t)$ contained in each domain $\mathcal{F}_k (t)$ evolve according to the law:

$$V_n = -c(v).$$

This provides some coupling between $\mathcal{U}^k_+ (t), \mathcal{U}^k_- (t)$ and $v$. Actually the spiral waves that we have obtained with the ODEs are just the solutions under this coupling.

(III) The coupling between the domains $\mathcal{U}^k_+ (t), \mathcal{U}^k_- (t)$ and the one of $V$ comes through the Young contact angle at the triple junctions connecting $\partial \mathcal{U}^k_+ (t), \partial \mathcal{U}^k_- (t)$ and $\gamma_{i,j}$. The corresponding interfacial energy depends on the specific shape of $V(x,\cdot,t)$. On the other hand, stability considerations indicate that the triple junctions where $\mathcal{U}^k_- (t)$ invades $\gamma_{i,j}$ have $v = v^*$, at least if one of the sides of the interface we have a region of the form $u = u_s, v = v_s$ (i.e. $u = U_- = u_s$).
Conclusions: scenario suggested by the results of this paper.

The asymptotics and theorems obtained in this paper suggest the existence of solutions of the Fitz-Hugh Nagumo system (1.1) for small \( \varepsilon \) with the behavior described in Section 8. It is not clear if the solutions described in this paper using asymptotic ideas exist for arbitrarily long times, or just for some large intervals depending in \( \varepsilon \). The existence of the asymptotic solutions described in this paper for very long intervals is plausible because the change of solutions in the layers described by the "transition problem" contained in Section 7 affects only very small regions, and it does not have time to modify the structure of the spirals in a meaningful way.

If the triple junction determines a unique angle as suggested, this would determine a unique value of the radius \( R \) for each each \( \omega \). Therefore the results of this paper would suggest the existence of a one-parameter family of long-lived solutions of (1.1) behaving as described in Section 8. Additional research is needed to determine if a selection of \( \omega \) is possible for specific nonlinearities, as well as in order to clarify the behavior of the solutions near the triple junction, and the study of the stability properties of the solutions near the back of the wave. Also numerical computations would possibly shed some light on the existence of a selection mechanism to fix a given frequency for stable spirals. However, the sharp transitions involved in the description proposed in this paper for both variables, \( u \) and \( v \), require a highly non trivial numerical treatment due to the stiffness of the problem.

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A Non existence of rigidly rotating spiral waves with \( v \) continuous everywhere.

We assume the usual conditions (A1)-(A6) for the functions \( f, g \).

It is interesting to remark that, in the previously described model it is not possible to obtain spiral waves rotating at constant angular speed, without discontinuities for \( v \). Indeed, suppose that there exists a unique domain \( F_1(t) = \mathbb{R}^2 \). Let us denote the interface separating \( U_+ (t) \), \( U_- (t) \) as \( \Gamma (t) \). By assumption:

\[
\Gamma (t) = \mathcal{R}_\omega t (\Gamma (0)) \quad , \quad \omega > 0.
\]

We now have two possibilities. Either \( 0 \in \Gamma (0) \) or \( 0 \notin \Gamma (0) \). Suppose first that \( 0 \notin \Gamma (0) \). Let us denote by \( P \) the point in \( \Gamma (0) \) closest to the origin. Let us denote as \( \rho \) the distance between the origin and \( P \). Notice that the disc \( B_\rho (0) \) is then completely contained either in \( U_+ (t) \) or \( U_- (t) \). Since \( g(U_+(v), v) > 0 \) in the whole range of admissible \( v's \), it turns out that the first case would yield an unbounded \( v \) and therefore a contradiction. On the other hand, if \( 0 \in U_- (t) \) in a rotating coordinate system we have:

\[
- \omega \frac{dv}{d\theta} (r, \theta) = g(U_-(v), v).
\]
This is a first order ODE and the solution \( v(r, \cdot) \) is periodic with period \( 2\pi \). Then, the only possibility \( v = v_{\text{X}} \), that is the only zero of \( g(U_{\pm}(v), v) \) in the admissible interval of values of \( v \).

Since \( v \) is a continuous function, it follows that \( v(P) = v_{\text{X}} \). Since \( P \) is the closest point to the origin, it follows that the normal vector to \( \Gamma(0) \) at \( P \) is parallel to the vector connecting 0 and \( P \). Therefore, the normal velocity at \( R_{\text{ad}}(P) \), \( V_n(R_{\text{ad}}(P), t) = 0 \), whence:

\[
c(v_{\text{X}}) = 0,
\]

but this contradicts our assumptions on \( f, g \).

Suppose then that \( 0 \in \Gamma(0) \). The normal velocity \( V_n(0, t) = 0 \), whence \( v(0, t) = v^* \). By assumption \( g(U_+(v^*), v^*) > 0 \), \( g(U_-(v^*), v^*) < 0 \). A local analysis of the equation \( v_t = g(U_{\pm}(v), v) \) implies that \( v \) has a variation of order one for points \( x \) arbitrarily close to the origin. Therefore \( v \) is not a continuous function, against our hypothesis. This concludes the proof.

### B Onset of discontinuities for \( v \) for the sharp interface limit.

One of the key features of this paper is the fact that \( v \) is discontinuous along some curves in the limit \( \varepsilon = 0 \). It turns out that such a regions of discontinuity can arise starting from initially smooth distributions. The one-dimensional sharp interface limit is the following. We assume that the real line \( \mathbb{R} \) is decomposed in a family of open intervals where \( u = U_+ \) or \( u = U_- \). These intervals are separated by points \( X = X(t) \) evolving by means of the equation:

\[
\frac{dX}{dt} = c(v(X, t)).
\]

On the other hand the function \( v(x, t) \), that will be assumed to be initially continuous, evolves according to the equation:

\[
\frac{\partial v(x, t)}{\partial t} = g(U_+(v(x, t)), v(x, t)),
\]

where the choice \( U_{\pm} \) depends on the region.

Let us consider the following particular configuration. Suppose that there is an interface at the point \( x = X(t) \), separating the regions \( x < X(t) \) where \( u = U_+ \) and \( x > X(t) \) where \( u = U_- \). We assume also that \( v_0 = v(x, 0) \) is a decreasing function satisfying \( v_0(x^*) = v^* \) at some \( x^* > X(0) \). Therefore, initially, at the points \( x > X(t) \) we have:

\[
\frac{\partial v(x, t)}{\partial t} = g(U_-(v(x, t)), v(x, t)) < 0.
\]

On the other hand, as the point \( X(t) \) approaches the point \( x^*(t) \) where \( v(x^*(t), t) = v^* \) the speed \( c(v(X(t), t)) \) approaches zero. Since \( g(U_-(v(x, t)), v(x, t)) \leq -\theta < 0 \) it then follows that at some finite time \( t^* \) we have:

\[
v(X(t^*), t^*) = v^*.
\]

It then follows that for \( x < X(t^*) \) we have \( v_t > 0 \) and for \( x > X(t^*) \) we would have \( v_t < 0 \). Since both derivatives are of order one this would generate a discontinuity for \( v \) instantaneously.

Notice that other possible dynamics are not possible. Indeed, suppose that the point \( x^*(t) \) moves to the left of \( X(t) \) for \( t > t^* \). In such a case \( v_t(X(t), t) > 0 \), and \( c(v(X, t)) \) would be small. This would shift the position of the point \( x^*(t) \) towards \( X(t) \). The same would happen if \( x^*(t) \) moves to the right of \( X(t) \) for \( t > t^* \). Therefore, the only reasonable evolution that can take place for \( v, X \), is the formation of a discontinuity for \( v \).
References


