# Compositional Time Series: A First Approach

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**Abstract:** Compositional time series, i.e., multivariate time series of vectors of $D$ proportions, arise in many areas of application where the focus of attention is on the relative, rather than the absolute, values of their components. Such series are characterized by components which are positive and sum to one at each instance in time. Although data of this type constitute multivariate time series, standard modelling techniques are not applicable due to the positivity of the components and the constant sum constraint. In other words, problems arise because its sample space is not the $D$-dimensional real space, nor the positive real space, but the $(D-1)$-dimensional simplex space. We consider basic concepts regarding the Euclidean structure of the simplex, and the alr, clr and ilr transformations on it are introduced to present compositional ARIMA models.

**Keywords:** ARIMA models, Compositional time series, Simplex

## 1 The simplex $\mathcal{S}^D$ as a compositional space

### 1.1 The simplex as a real vector space

A $D$-part composition $\mathbf{x} = (x_1, \ldots, x_D)'$ is any element of the simplex

$$\mathcal{S}^D = \{(x_1, \ldots, x_D)' : x_1 > 0, \ldots, x_D > 0; x_1 + \ldots + x_D = 1\}.$$

Basic operations on $\mathcal{S}^D$ have been introduced by Aitchison (1986) and Barceló-Vidal et al (2002). The *perturbation* operation is defined as

$$\mathbf{x} \oplus \mathbf{x}^* = \mathcal{C}(x_1 x_1^*, \ldots, x_D x_D^*)' \text{ for any } \mathbf{x}, \mathbf{x}^* \in \mathcal{S}^D,$$

and the *power transformation*, defined for any $\mathbf{x} \in \mathcal{S}^D$ and $\alpha \in \mathbb{R}$ as

$$\alpha \odot \mathbf{x} = \mathcal{C}(x_1^\alpha, \ldots, x_D^\alpha)',$$

where $\mathcal{C}$ denotes the *closure* operator defined for any $\mathbf{z} \in \mathbb{R}_+^D$ as $\mathcal{C}\mathbf{z} = \mathbf{z}/\sum_{i=1}^D z_i$. In this manner $(\mathcal{S}^D, \oplus, \odot)$ becomes a real vector space of dimension $D-1$. The composition $\mathbf{0}_\mathcal{C} = (1/D, \ldots, 1/D)'$ is the neutral (zero) element, and the inverse (opposite) of $\mathbf{x} \in \mathcal{S}^D$ is the composition $\mathbf{x}^{-1} = \mathcal{C}(1/x_1, \ldots, 1/x_D)'$.

Provided that $(\mathcal{S}^D, \oplus, \odot)$ is a real vector space, it can be viewed as an affine space when the group $(\mathcal{S}^D, \oplus)$ operates on $\mathcal{S}^D$ as a group of transformations. Perturbations in the compositional space plays the same role as translations plays in the real space. The assumption that the group of perturbations is the operating group on the compositional space is the keystone of the methodology introduced by Aitchison (1986). In fact, it means accepting that the "difference" between two compositions $\mathbf{x}$ and $\mathbf{x}^*$ is the composition $\mathbf{x} \ominus \mathbf{x}^* = \mathcal{C}(x_1/x_1^*, \ldots, x_D/x_D^*)'$, based on the ratios $x_j/x_j^*$ between parts instead of on the subtraction $x_j^* - x_j$.

### 1.2 Transformations on the simplex

Let $\mathcal{A}_{D\times D}$ denote the family of all real $D \times D$ matrices such that $\mathbf{A}\mathbf{1}_D = \mathbf{A}'\mathbf{1}_D = \mathbf{0}_D$. Let $\mathbf{x} \in \mathcal{S}^D$ and $\mathbf{A} \in \mathcal{A}_{D\times D}$. We define the *product* $\mathbf{A} \odot \mathbf{x}$ as

$$\mathbf{A} \odot \mathbf{x} = \mathcal{C}\left(\prod_{j=1}^{D} x_j^{a_{1j}}, \ldots, \prod_{j=1}^{D} x_j^{a_{Dj}}\right)'.$$

The function $\mathbf{x} \rightarrow \mathbf{A} \odot \mathbf{x}$ is an endomorphism of the vector space $(\mathcal{S}^D, \oplus, \odot)$. Moreover, any endomorphism of $\mathcal{S}^D$ can be written in this form. The matrix associated to identity endomorphism is the well-known *centering matrix* $\mathbf{G}_D = \mathbf{I}_D - D^{-1}\mathbf{J}_D$ of order $D \times D$.

The *additive logratio transformation* of index $j$ $(j = 1, \ldots, D)$ —denoted by $\mathrm{alr}_j$— is the one-to-one transformation from $\mathcal{S}^D$ to $\mathbb{R}^{D-1}$ defined as $\mathbf{x} \longrightarrow \mathbf{y} = \mathrm{alr}_j\, \mathbf{x} = \log(\mathbf{x}_{-j}/x_j)$ where $\mathbf{x}_{-j}$ denotes the vector $\mathbf{x}$ with the component $x_j$ deleted. In particular, we use alr —without any subindex— to denote the transformation $\mathrm{alr}_D$. The inverse transformation of $\mathrm{alr}_j$ is the well known *additive logistic transformation*.

The *centered* (or *symmetric*) *logratio transformation* —denoted by clr— is the function from the compositional space $\mathcal{S}^D$ to $\mathbb{R}^D$, defined by $\mathbf{x} \longrightarrow \mathbf{z} = \mathrm{clr}\, \mathbf{x} = \log(\mathbf{x}/g(\mathbf{x}))$, where $g(\mathbf{x})$ is the geometric mean of the components of $\mathbf{x}$, i.e., $g(\mathbf{x}) = (x_1 x_2 \ldots x_D)^{1/D}$. This transformation maps $\mathcal{S}^D$ in the subspace $V = \{\mathbf{z} \in \mathbb{R}^D : z_1 + \ldots + z_D = 0\}$ of $\mathbb{R}^D$, which can be seen to be a hyperplane through the origin of $\mathbb{R}^D$, orthogonal to $\mathbf{1}_D$ (vector of units). This subspace has dimension $D-1$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_{D-1}$ be any orthonormal basis of $V$, and let $\mathbf{V}$ be the $D \times (D-1)$ matrix $[\mathbf{v}_1 : \ldots : \mathbf{v}_{D-1}]$. Then, the *isometric logratio transformation* —denoted by $\mathrm{ilr}_V$— associated with this matrix $\mathbf{V}$, is the one-to-one transformation from $\mathcal{S}^D$ to $\mathbb{R}^{D-1}$ which assigns to each composition $\mathbf{x}$ the components of $\mathrm{clr}\, \mathbf{x}$ in the basis $\mathbf{v}_1, \ldots, \mathbf{v}_{D-1}$ of $V$. It can be proved that $\mathbf{x} \longrightarrow \mathbf{u} = \mathrm{ilr}_V \mathbf{x} = (\mathbf{F}\mathbf{V})^{-1}\mathbf{F} \log \mathbf{x}$, for any $\mathbf{x} \in \mathcal{S}^D$, where $\mathbf{F}$ is the $(D-1) \times D$ matrix $[\mathbf{I}_{D-1} : -\mathbf{1}_{D-1}]$.

It is very important to emphasize that all these transformations —$\mathrm{alr}_j$, clr, $\mathrm{ilr}_V$, and its inverses— are one-to-one linear transformations between the compositional vector space $(\mathcal{S}^D, \oplus, \odot)$ and the real vector space $(\mathbb{R}^{D-1}, +, .)$

(or $V \subset \mathbb{R}^D$) with the natural structure. Vectors $\mathbf{u} = \mathrm{ilr}_V\,\mathbf{x}$, $\mathbf{y} = \mathrm{alr}_D\,\mathbf{x}$ and $\mathbf{z} = \mathrm{clr}\,\mathbf{x}$ associated with the same composition $\mathbf{x} \in \mathcal{S}^D$ are related by the following linear relationships expressed in matrix form:

1. $\mathbf{u} = (\mathbf{FV})^{-1}\mathbf{y}$, and $\mathbf{u} = (\mathbf{FV})^{-1}\mathbf{Fz}$.
2. $\mathbf{y} = \mathbf{FVu}$, and $\mathbf{y} = \mathbf{Fz}$.
3. $\mathbf{z} = \left((\mathbf{FV})^{-1}\mathbf{F}\right)'\mathbf{u}$, and $\mathbf{z} = \mathbf{F}'\mathbf{H}^{-1}\mathbf{y}$, where $\mathbf{H}$ is the $(D-1) \times (D-1)$ matrix $\mathbf{FF}' = \mathbf{I}_{D-1} + \mathbf{J}_{D-1}$, with $\mathbf{J}_{D-1} = \mathbf{1}_{D-1}\mathbf{1}'_{D-1}$.

### 1.3 The simplex as a metric space

The one-to-one linear transformation clr allows one to transfer the real Euclidean structure defined on $\mathbb{R}^{D-1}$ to $\mathcal{S}^D$ . Thus the compositional norm ($\mathcal{C}$-norm) of $\mathbf{x} \in \mathcal{S}^D$ is equal to the Euclidean norm in $\mathbb{R}^D$ of the clr-transformed vector, i.e., $\|\mathbf{x}\|_\mathcal{C} = \|\mathrm{clr}\,\mathbf{x}\|$, and the $\mathcal{C}$-distance between two compositions $\mathbf{x}, \mathbf{x}^* \in \mathcal{S}^D$ is given by the $\mathcal{C}$-norm of the difference $\mathbf{x} \ominus \mathbf{x}^*$. Thus the $\mathcal{C}$-distance just defined converts $\mathcal{S}^D$ into a Euclidean space and the transformation clr is the natural isometry between $\mathcal{S}^D$ and the subspace $V$ of $\mathbb{R}^D$. Moreover, as the $D-1$ columns of matrix $\mathbf{V}$ used in the transformation $\mathrm{ilr}_V$ constitute, by definition, an orthonormal basis of $V$, this transformation is also an isometry between $\mathcal{S}^D$ and $\mathbb{R}^{D-1}$. The same cannot be said for the additive logratio transformations $\mathrm{alr}_j$.

### 1.4 The covariance structure of the simplex

Let $\mathbf{x}$ be a random $D$-part composition defined on $\mathcal{S}^D$. According to the metric structure defined on $\mathcal{S}^D$, the $\mathcal{C}$-mean of $\mathbf{x}$, symbolized by $\boldsymbol{\xi}$ or $\mathrm{E}_\mathcal{C}\{\mathbf{x}\}$ , is defined as $\boldsymbol{\xi} = \mathrm{clr}^{-1}\mathrm{E}\{\mathrm{clr}\,\mathbf{x}\}$ and the $\mathcal{C}$-covariance matrix $\boldsymbol{\Sigma}^\mathcal{C}$ of $\mathbf{x}$ as

$$\boldsymbol{\Sigma}^\mathcal{C} = \left[\mathrm{cov}\left\{\log\frac{x_i}{g(\mathbf{x})}, \log\frac{x_j}{g(\mathbf{x})}\right\}\right] = \left[\sigma_{ij}^\mathcal{C}\right]_{i,j=1}^D,$$

i.e., by the covariance matrix $\boldsymbol{\Sigma}^\mathcal{Z}$ of the random vector $\mathbf{z} = \mathrm{clr}\,\mathbf{x}$, known as *centred logratio matrix*. The consequent singularity of the distribution $\mathbf{z} = \mathrm{clr}\,\mathbf{x}$ is reflected in the singularity of $\boldsymbol{\Sigma}^\mathcal{C}$, since $\boldsymbol{\Sigma}^\mathcal{C}\mathbf{1}_D = \mathbf{0}_D$.
Aitchison (1986) defines other matrices to determine the $\mathcal{C}$-covariance structure of a random composition $\mathbf{x}$. The *variation matrix* $\mathbf{T}$ is defined as

$$\mathbf{T} = \left[\mathrm{var}\left\{\log\left(x_i/x_j\right)\right\}\right]_{i,j=1}^D = \left[\tau_{ij}\right]_{i,j=1}^D,$$

and the *logratio covariance matrix* $\boldsymbol{\Sigma}^\mathcal{Y}$ as

$$\boldsymbol{\Sigma}^\mathcal{Y} = \left[\mathrm{cov}\left\{\log\frac{x_i}{x_D}, \log\frac{x_j}{x_D}\right\}\right]_{i,j=1}^{D-1} = \left[\sigma_{ij}^\mathcal{Y}\right]_{i,j=1}^{D-1},$$

i.e., by the covariance matrix of the random vector $\mathbf{y} = \text{alr}\,\mathbf{x}$ on $\mathbb{R}^{D-1}$. It is clear that $\boldsymbol{\Sigma}^{\mathcal{Y}}$ will depend on the denominator used in the alr-transformation. Finally, the covariance matrix of the random vector $\mathbf{u} = \text{ilr}\,\mathbf{x}$ on $\mathbb{R}^{D-1}$ will be denoted by $\boldsymbol{\Sigma}^{\mathcal{U}} = \left[\sigma_{ij}^{\mathcal{U}}\right]_{i,j=1}^{D-1}$. This covariance matrix will depend on the matrix $\mathbf{V}$ used in the ilr-transformation. Although the $\mathcal{C}$-covariance structure of $\mathbf{x}$ is given by $\boldsymbol{\Sigma}^{\mathcal{C}}$, the relationships between all these matrices allow one to deduce $\boldsymbol{\Sigma}^{\mathcal{C}}$ from any of the other matrices.

### 1.5 Joint distribution on the simplex

Let $(\mathbf{x}_1, \mathbf{x}_2)$ be a bivariate random compositional vector defined on $\mathcal{S}^D \times \mathcal{S}^D$. If $\boldsymbol{\xi}_i = \text{E}_{\mathcal{C}}\{\mathbf{x}_i\}$ $(i = 1, 2)$, the $\mathcal{C}$-covariance matrix $\boldsymbol{\Gamma}^{\mathcal{C}}(\mathbf{x}_1, \mathbf{x}_2) = \left[\gamma_{ij}^{\mathcal{C}}\right]_{i,j=1}^{D}$ of $(\mathbf{x}_1, \mathbf{x}_2)$ is defined as

$$\boldsymbol{\Gamma}^{\mathcal{C}}(\mathbf{x}_1, \mathbf{x}_2) = \left[\text{E}\left\{\left(\log\frac{x_{1i}}{g(\mathbf{x}_1)} - \log\frac{\xi_{1i}}{g(\boldsymbol{\xi}_1)}\right)\left(\log\frac{x_{2j}}{g(\mathbf{x}_2)} - \log\frac{\xi_{2j}}{g(\boldsymbol{\xi}_2)}\right)\right\}\right]_{i,j=1}^{D}.$$

Therefore, $\boldsymbol{\Gamma}^{\mathcal{C}}(\mathbf{x}_1, \mathbf{x}_2)$ coincides with the covariance matrix $\boldsymbol{\Gamma}^{\mathcal{Z}}(\mathbf{z}_1, \mathbf{z}_2)$ of $(\mathbf{z}_1, \mathbf{z}_2) = (\text{clr}\,\mathbf{x}_1, \text{clr}\,\mathbf{x}_2)$ defined on $V \times V \subset \mathbb{R}^D \times \mathbb{R}^D$. The matrix $\boldsymbol{\Gamma}^{\mathcal{C}}(\mathbf{x}_1, \mathbf{x}_2)$ is not symmetric but is singular because $\boldsymbol{\Gamma}^{\mathcal{C}}\mathbf{1}_D = (\boldsymbol{\Gamma}^{\mathcal{C}})'\mathbf{1}_D = \mathbf{0}_D$. We denote by $\boldsymbol{\Gamma}^{\mathcal{Y}}(\mathbf{y}_1, \mathbf{y}_2) = \left[\gamma_{ij}^{\mathcal{Y}}\right]_{i,j=1}^{D-1}$ the covariance matrix of $(\mathbf{y}_1, \mathbf{y}_2) = (\text{alr}\,\mathbf{x}_1, \text{alr}\,\mathbf{x}_2)$, and by $\boldsymbol{\Gamma}^{\mathcal{U}}(\mathbf{u}_1, \mathbf{u}_2) = \left[\gamma_{ij}^{\mathcal{U}}\right]_{i,j=1}^{D-1}$ the covariance matrix of $(\mathbf{u}_1, \mathbf{u}_2) = (\text{ilr}\,\mathbf{x}_1, \text{ilr}\,\mathbf{x}_2)$. As before, there exists matrix relationships between the covariance matrices $\boldsymbol{\Gamma}^{\mathcal{C}}$, $\boldsymbol{\Gamma}^{\mathcal{Y}}$ and $\boldsymbol{\Gamma}^{\mathcal{U}}$.

## 2 Compositional time series models

Let $\mathbf{x}_t = (x_{t1}, \ldots, x_{tD})'$, $t = 0, \pm1, \pm2, \ldots$ be a compositional process ($\mathcal{C}$-time series process) defined on $\mathcal{S}^D$ for any $t$. The compositional second-order properties of $\mathbf{x}_t$ are then specified by the $\mathcal{C}$-mean vectors, $\boldsymbol{\xi}_t = \text{E}_{\mathcal{C}}\{\mathbf{x}_t\} = (\xi_{t1}, \ldots, \xi_{tD})'$, and the $\mathcal{C}$-autocovariances matrices,

$$\boldsymbol{\Gamma}^{\mathcal{C}}(t{+}h, t) = \text{E}\left\{\left(\text{clr}\,\mathbf{x}_{t+h} - \text{clr}\,\boldsymbol{\xi}_{t+h}\right)\left(\text{clr}\,\mathbf{x}_t - \text{clr}\,\boldsymbol{\xi}_t\right)'\right\} = \left[\gamma_{ij}^{\mathcal{C}}(t+h, t)\right]_{i,j=1}^{D},$$

which belong to the family of $\mathcal{A}_{D\times D}$ matrices.

Notice that in the compositional context, given a $\mathcal{C}$-time series $\{\mathbf{x}_t\}$ it makes no sense to analyze any of the individual parts $\{x_{ti}\}$ as univariate time series. However, in some cases one might be interested in analyzing the relative behavior of two parts $i$ and $j$ $(i \neq j)$ or, in general, of a subcompositional time series $\{\mathbf{x}_{St}\}$, where $S$ symbolizes any subset of two or more parts $1, \ldots, D$ of $\mathbf{x}_t$.

The clr, alr and ilr transformations applied to a compositional process $\{\mathbf{x}_t\}$ induce three processes $\{\mathbf{z}_t\}$, $\{\mathbf{y}_t\}$ and $\{\mathbf{u}_t\}$, respectively. The former, $\{\mathbf{z}_t\}$, defined on $\mathbb{R}^D$, is restricted to the hyperplane $V$ because $\mathbf{z}_t'\mathbf{1}_D = 0$. The other two time series processes are defined on $\mathbb{R}^{D-1}$ but $\{\mathbf{y}_t\}$ depends on the denominator used in the alr-transformation and $\{\mathbf{u}_t\}$ on the matrix $\mathbf{V}$ used in the ilr-transformation. We denote by $\boldsymbol{\mu}_t^{\mathcal{Z}}, \boldsymbol{\mu}_t^{\mathcal{Y}}$ and $\boldsymbol{\mu}_t^{\mathcal{U}}$ the mean vectors of $\{\mathbf{z}_t\}$, $\{\mathbf{y}_t\}$ and $\{\mathbf{u}_t\}$, respectively, and by $\boldsymbol{\Gamma}^{\mathcal{Z}}(t+h,t), \boldsymbol{\Gamma}^{\mathcal{Y}}(t+h,t)$ and $\boldsymbol{\Gamma}^{\mathcal{U}}(t+h,t)$ the autocovariance matrices of these time series processes. Observe that, by definition, $\boldsymbol{\mu}_t^{\mathcal{Z}} = \operatorname{clr} \boldsymbol{\xi}_t$ and $\boldsymbol{\Gamma}^{\mathcal{Z}}(t+h,t) = \boldsymbol{\Gamma}^{\mathcal{C}}(t+h,t)$. The mean vectors $\boldsymbol{\mu}_t^{\mathcal{Y}}$ and $\boldsymbol{\mu}_t^{\mathcal{U}}$, and the covariance matrices $\boldsymbol{\Gamma}^{\mathcal{Y}}(t+h,t)$ and $\boldsymbol{\Gamma}^{\mathcal{U}}(t+h,t)$ are related to $\operatorname{clr} \boldsymbol{\xi}_t$ and $\boldsymbol{\Gamma}^{\mathcal{C}}(t+h,t)$ by the equations given in 2.2.

## 2.1 Stationary $\mathcal{C}$-time series processes

The $\mathcal{C}$-time series process $\{\mathbf{x}_t\}$ is said to be (weakly) $\mathcal{C}$*-stationary* if $\boldsymbol{\xi}_t$ and $\boldsymbol{\Gamma}^{\mathcal{C}}(t+h,t),\ h = 0, \pm 1, \ldots$ are independent of $t$. For a $\mathcal{C}$-stationary process we use the notation

$$\boldsymbol{\xi} = \mathrm{E}_{\mathcal{C}}\{\mathbf{x}_t\} \ ; \ \boldsymbol{\Gamma}^{\mathcal{C}}(h) = \mathrm{E}\left\{(\operatorname{clr} \mathbf{x}_{t+h} - \operatorname{clr} \boldsymbol{\xi})(\operatorname{clr} \mathbf{x}_t - \operatorname{clr} \boldsymbol{\xi})'\right\} = \left[\gamma_{ij}^{\mathcal{C}}(h)\right]_{i,j=1}^{D}.$$

We shall refer to $\boldsymbol{\xi}$ as the $\mathcal{C}$-mean of $\{\mathbf{x}_t\}$ and to $\boldsymbol{\Gamma}^{\mathcal{C}}(h)$ as the $\mathcal{C}$-autocovariance at lag $h$, and $\boldsymbol{\Gamma}^{\mathcal{C}}(h)_{h=0,1,\ldots}$ as the $\mathcal{C}$-autocovariance function. The $\mathcal{C}$-autocorrelation function $\mathbf{R}(h)_{h=0,1,\ldots}$ is defined by

$$\mathbf{R}^{\mathcal{C}}(h) = \left[\gamma_{ij}^{\mathcal{C}}(h) / \sqrt{\gamma_{ii}^{\mathcal{C}}(0)\gamma_{jj}^{\mathcal{C}}(0)}\right]_{i,j=1}^{D} = \left[\rho_{ij}^{\mathcal{C}}(h)\right]_{i,j=1}^{D}.$$

The $\mathcal{C}$-time series process $\{\mathbf{w}_t\}$ is said to be $\mathcal{C}$*-white noise* with $\mathcal{C}$-mean $\mathbf{0}_{\mathcal{C}} = (1/D, \ldots, 1/D)'$ and $\mathcal{C}$-covariance matrix $\boldsymbol{\Sigma}^{\mathcal{C}}$ —written as $\{\mathbf{w}_t\} \sim \mathrm{WN}^{\mathcal{C}}(\mathbf{0}_{\mathcal{C}}, \boldsymbol{\Sigma}^{\mathcal{C}})$— if and only if $\{\mathbf{w}_t\}$ is $\mathcal{C}$-stationary with $\mathcal{C}$-mean vector $\mathbf{0}_{\mathcal{C}}$ and $\mathcal{C}$-autocovariance function

$$\boldsymbol{\Gamma}^{\mathcal{C}}(0) = \boldsymbol{\Sigma}^{\mathcal{C}}; \quad \boldsymbol{\Gamma}^{\mathcal{C}}(h) = \mathbf{0}_{D\times D}, \text{ if } h \neq 0.$$

The $\mathcal{C}$-stationary property of $\{\mathbf{x}_t\}$ is equivalent to the stationary property of any of the transformed processes $\{\mathbf{z}_t\}$, $\{\mathbf{y}_t\}$ and $\{\mathbf{u}_t\}$. Moreover, $\{\mathbf{x}_t\}$ is $\mathcal{C}$-white noise if and only if $\{\mathbf{z}_t\}$ —or $\{\mathbf{y}_t\}$, or $\{\mathbf{u}_t\}$— is white noise.

## 2.2 $\mathcal{C}$-ARIMA processes

A $\mathcal{S}^D$-variate $\mathcal{C}$-time series process $\{\mathbf{x}_t\}$ is a $\mathcal{C}$-ARMA$(p,q)$ process if

$$(\mathbf{x}_t \ominus \boldsymbol{\xi}) \ominus (\boldsymbol{\Phi}_1 \odot (\mathbf{x}_{t-1} \ominus \boldsymbol{\xi})) \ominus \ldots \ominus (\boldsymbol{\Phi}_p \odot (\mathbf{x}_{t-p} \ominus \boldsymbol{\xi})) = \mathbf{w}_t \ominus (\boldsymbol{\Theta}_1 \odot \mathbf{w}_{t-1}) \ominus \ldots \ominus (\boldsymbol{\Theta}_q \odot \mathbf{w}_{t-q}),$$

where $\mathbf{\Phi}_1,\ldots,\mathbf{\Phi}_p,\mathbf{\Theta}_1,\ldots,\mathbf{\Theta}_q$ are $\mathcal{A}_{D\times D}$-matrices and $\mathbf{w}_t \sim \mathrm{WN}^{\mathcal{C}}(\mathbf{0}_{\mathcal{C}},\mathbf{\Sigma}^{\mathcal{C}})$. These equations can be written in the more compact form

$$\mathbf{\Phi}^{\mathcal{C}}(L_{\mathcal{C}})(\mathbf{x}_t \ominus \boldsymbol{\xi}) = \mathbf{\Theta}^{\mathcal{C}}(L_{\mathcal{C}})\mathbf{w}_t, \quad \{\mathbf{w}_t\} \sim \mathrm{WN}^{\mathcal{C}}(\mathbf{0}_{\mathcal{C}},\mathbf{\Sigma}^{\mathcal{C}}),$$

where $\mathbf{\Phi}^{\mathcal{C}}(z) = \mathbf{G}_D \ominus (\mathbf{\Phi}_1 \odot z) \ominus \ldots \ominus (\mathbf{\Phi}_p \odot z^p)$ and $\mathbf{\Theta}^{\mathcal{C}}(z) = \mathbf{G}_D \ominus (\mathbf{\Theta}_1 \odot z) \ominus \ldots \ominus (\mathbf{\Theta}_q \odot z^q)$ are $\mathcal{A}_{D\times D}$-matrix-valued polynomials, $\mathbf{G}_D$ is the centering matrix and $L_{\mathcal{C}}$ the backshift operator. In the compositional context, the operator $1-L_{\mathcal{C}}$ represents the $\mathcal{C}$-difference operator, i.e., $(1-L_{\mathcal{C}})\mathbf{x}_t = \mathbf{x}_t \ominus \mathbf{x}_{t-1}$. Applying $1-L_{\mathcal{C}}$ to $\{\mathbf{x}_t\}$ is equivalent to apply $1-L$ to the transformed processes $\{\mathbf{z}_t\}$, $\{\mathbf{y}_t\}$ and $\{\mathbf{u}_t\}$.

If $\{\mathbf{x}_t\}$ is $\mathcal{C}$-ARMA$(p,q)$ process then $\{\mathbf{z}_t\}$ is an ARMA$(p,q)$ process because

$$\mathbf{\Phi}^{\mathcal{Z}}(L)\left(\mathbf{z}_t - \boldsymbol{\mu}^{\mathcal{Z}}\right) = \mathbf{\Theta}^{\mathcal{Z}}(L)\mathbf{w}_t^{\mathcal{Z}}, \quad \{\mathbf{w}_t^{\mathcal{Z}}\} \sim \mathrm{WN}(\mathbf{0}_D,\mathbf{\Sigma}^{\mathcal{Z}}),$$

where $\mathbf{\Phi}^{\mathcal{Z}}(z) = \mathbf{I}_D - \sum_{i=1}^{p}\mathbf{\Phi}_i z^i$; $\mathbf{\Theta}^{\mathcal{Z}}(z) = \mathbf{I}_D - \left(\sum_{i=1}^{q}\mathbf{\Theta}_i z^i\right)$; and $\mathbf{\Sigma}^{\mathcal{Z}} = \mathbf{\Sigma}^{\mathcal{C}}$. Equally, $\{\mathbf{y}_t\}$ will be an ARMA$(p,q)$ process because

$$\mathbf{\Phi}^{\mathcal{Y}}(L)\left(\mathbf{y}_t - \boldsymbol{\mu}^{\mathcal{Y}}\right) = \mathbf{\Theta}^{\mathcal{Y}}(L)\mathbf{w}_t^{\mathcal{Y}}, \quad \{\mathbf{w}_t^{\mathcal{Y}}\} \sim \mathrm{WN}(\mathbf{0}_{D-1},\mathbf{\Sigma}^{\mathcal{Y}}),$$

where

$$\mathbf{\Phi}^{\mathcal{Y}}(z) = \mathbf{I}_{D-1} - \left(\sum_{i=1}^{p}\mathbf{F}\mathbf{\Phi}_i\mathbf{F}'\mathbf{H}^{-1}z^i\right), \quad \mathbf{\Theta}^{\mathcal{Y}}(z) = \mathbf{I}_{D-1} - \left(\sum_{i=1}^{q}\mathbf{F}\mathbf{\Theta}_i\mathbf{F}'\mathbf{H}^{-1}z^i\right)$$

and $\mathbf{\Sigma}^{\mathcal{Y}} = \mathbf{F}\mathbf{\Sigma}^{\mathcal{C}}\mathbf{F}'$. And $\{\mathbf{u}_t\}$ will be an ARMA$(p,q)$ process because

$$\mathbf{\Phi}^{\mathcal{U}}(L)\left(\mathbf{u}_t - \boldsymbol{\mu}^{\mathcal{U}}\right) = \mathbf{\Theta}^{\mathcal{U}}(L)\mathbf{w}_t^{\mathcal{U}}, \quad \{\mathbf{w}_t^{\mathcal{U}}\} \sim \mathrm{WN}(\mathbf{0}_{D-1},\mathbf{\Sigma}^{\mathcal{U}}),$$

where $\mathbf{\Phi}^{\mathcal{U}}(z) = \mathbf{I}_{D-1} - \left(\sum_{i=1}^{p}\mathbf{U}'\mathbf{\Phi}_i\mathbf{U}z^i\right)$; $\mathbf{\Theta}^{\mathcal{U}}(z) = \mathbf{I}_{D-1} - \left(\sum_{i=1}^{q}\mathbf{U}'\mathbf{\Theta}_i\mathbf{U}z^i\right)$ —with $\mathbf{U} = \mathbf{F}'\mathbf{H}^{-1}\mathbf{F}\mathbf{V}$—; and $\mathbf{\Sigma}^{\mathcal{U}} = (\mathbf{F}\mathbf{V})^{-1}\mathbf{F}\mathbf{\Sigma}^{\mathcal{C}}\left((\mathbf{F}\mathbf{V})^{-1}\mathbf{F}\right)'$.

If $d$ is a non-negative integer, it is natural to define $\{\mathbf{x}_t\}$ as a $\mathcal{C}$-ARIMA$(p,d,q)$ processes if $(1-L_{\mathcal{C}})^d\mathbf{x}_t$ is a $\mathcal{C}$-ARMA$(p,q)$ processes. This definition means that $\{\mathbf{x}_t\}$ satisfies a $\mathcal{C}$-difference equation of the form

$$\mathbf{\Phi}^{\mathcal{C}}(L_{\mathcal{C}})(1-L_{\mathcal{C}})^d\mathbf{x}_t = \mathbf{\Theta}^{\mathcal{C}}(L_{\mathcal{C}})\mathbf{w}_t, \quad \{\mathbf{w}_t\} \sim \mathrm{WN}^{\mathcal{C}}(\mathbf{0}_{\mathcal{C}},\mathbf{\Sigma}^{\mathcal{C}}),$$

where $\mathbf{\Phi}^{\mathcal{C}}(z) = \mathbf{G}_D \ominus (\mathbf{\Phi}_1 \odot z) \ominus \ldots \ominus (\mathbf{\Phi}_p \odot z^p)$ and $\mathbf{\Theta}^{\mathcal{C}}(z) = \mathbf{G}_D \ominus (\mathbf{\Theta}_1 \odot z) \ominus \ldots \ominus (\mathbf{\Theta}_q \odot z^q)$ are $\mathcal{A}_{D\times D}$-matrix-valued polynomials.

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