Lecture 8

Thermodynamics in Astrophysics and Cosmology

The concepts of thermodynamics have very wide applicability, including some rather unexpected ones. In this lecture I discuss two specific applications of thermodynamics to astrophysics and cosmology: the cosmic microwave background (CMB) radiation and the thermodynamics of black holes.

8.1 Big Bang cosmology

Cosmology is the study of the Universe on a large scale. Modern cosmology started with two developments early this century.

The first development was the formulation of general theory of relativity by Einstein in 1917. General relativity is a relativistic generalization of Newton's theory of gravitation and it quickly became the accepted theory for gravitation. Einstein himself was well aware of the cosmological implications of his theory, and included a term in his equations to take account of these implications. This term, which he called the cosmological constant, effectively implies that gravity has a repulsive term that is important on very large scales, of order the size of the Universe, but which has entirely negligible effects on much smaller scales. At that time it was thought that the Universe must be in a static state. If this were the case, then the attractive effect of gravity would cause the Universe as a whole to collapse. To prevent this otherwise inevitable collapse due to the self-gravity of the matter in the Universe Einstein introduced this repulsive term into the gravitational field equations, and this achieved the desired result of allowing a static solution for the Universe. Later, after the discovery that the Universe is expanding and is not is a static state, Einstein described the introduction of the cosmological constant as his "greatest mistake." Within a few years of the formulation of general relativity de Sitter formulated a model for the Universe in which the cosmological constant is zero and the Universe is expanding. The de Sitter model remains important in modern cosmology.

The other development was the observation by Hubble in the 1920s that the distant galaxies all appear to be receding from us, with a speed, v, proportional to their distance, r, from us. The constant of proportionality, H,

$$v = Hr, \tag{8.1}$$

is called the Hubble constant. It was immediately obvious that if the Universe is expanding then the simplest interpretation is that is had a beginning and so has a finite age. The Hubble constant has the dimensions of an inverse time, and the time H^{-1} must be characteristic of the age of the Universe. Measurement of the Hubble constant has been one of the objectives of cosmological research for several decades. The currently favored value is $\sim 70\,\mathrm{km\,s^{-1}/Mpc}$, where a megaparsec is $3.09 \times 10^{22}\,\mathrm{m}$. One finds $H^{-1} \sim 15 \times 10^{9}\,\mathrm{yr}$.

These ideas on the expanding Universe became the basis for cosmological theory as it developed over the subsequent several decades. However, there was little other direct evidence in favor of the basic model. This changed dramatically in 1965 with the accidental discovery of the CMB. Unbeknown to the discoverers, the CMB had been predicted by cosmologists, and at the time of the discovery another group had initiated an observational program to find the CMB. The discovery gave dramatic confirmation of the basic model of an expanding Universe. The prediction that was confirmed is based on the assumption that as the Universe expands, it cools adiabatically. As we trace back from the present, the Universe must has been hot enough at some time in the past for the hydrogen to be ionized. Hydrogen ionizes at a temperature of order $10^4\,\mathrm{K}$ (the actual temperature also depends on the density). Conversely, if we think of the Universe expanding from the Big Bang, then in an early phase all the matter is ionized, and when the Universe cools to $10^4\,\mathrm{K}$ the hydrogen recombines. Simple estimates imply that a typical photon in the black body spectrum at this 'recombination epoch' does not experience any further interaction with matter in the subsequent evolution of the Universe to the present day. Hence, there should be a relic spectrum of photons that corresponds to the adiabatically cooled thermal spectrum from the time when the temperature of the Universe was $\sim 10^4\,\mathrm{K}$. It is this relic radiation that is identified as the CMB.

8.2 The cosmological redshift

There is one fundamental idea in cosmology that needs to be understood before we discuss the CMB itself. This is the cosmological redshift, conventionally denoted by z. The quantity observed by Hubble, and interpreted in terms of an expansion velocity, is the redshift of spectral lines. The redshift is an observable parameter, but its interpretation is model dependent. When a redshifted line is observed, the relation between the wavelength of the observed (obs) radiation and the emitted (em) radiation is

$$\lambda_{\rm obs} = \frac{\lambda_{\rm em}}{1+z}. \tag{8.2}$$

If the redshift is interpreted as a Doppler shift due to a motion of the source away from us at a speed v, then for $z \ll 1$ we have $v = cz$. In modern astrophysics, observations of galaxies and quasars at $z > 5$ is now possible, and for $z > 1$ it is clear that the interpretation in terms of recessional velocity implies speeds that are highly relativistic.

The interpretation in terms of a Doppler shift, corresponding to a recession velocity, is not the most useful way to think of the cosmological redshift. A clearer interpretation is in terms of the scale size of the Universe. As the Universe expands all lengths expand. We can then make a model-independent statement: a cosmological redshift z implies that the photon was emitted at an epoch when the scale size of the Universe was $1/(1+z)$ of its present size. Thus $z = 1$ implies emission when the Universe was half its present size. The conversion of $1+z$ into distance (to the source where the photon was emitted) and time (the epoch when the photon was emitted) are both dependent on the details of the model of the Universe. However, the interpretation in terms of the scale size of the Universe is not dependent on the detailed model.

8.3 The Cosmic Microwave Background

Observations of the CMB have established that it is an excellent fit to a black body spectrum at a temperature $T = 2.73\,\mathrm{K}$. The prediction of the temperature is straightforward under the assumption that the expansion is adiabatic. A gas of photons obeys the adiabatic law with an adiabatic index of $\Gamma = 4/3$. The adiabatic law PV^Γ = constant may be rewritten in terms of T, V using the perfect gas law, and it implies $V^{\Gamma-1}T$ = constant. The volume varies as the scale size of the Universe cubed, so that we have $V \propto (1+z)^{-3}$, and with $\Gamma = 4/3$ we find $T/(1+z)$ = constant for black body radiation. In contrast to the radiation, the matter in the Universe cools adiabatically with an index $\gamma = 5/3$, implying that its temperature satisfies $T_{\rm matter}/(1+z)^2$ = constant. Thus the matter cools must faster than the radiation.

The details of when the radiation decouples from the matter involves a relatively lengthy calculation. The result for the redshift of the last scattering of a typical CMB photon gives $z \sim 1000$. Hence, observations of the CMB provides us with direct evidence on the properties of the Universe when the redshift was of order a thousand.

We assume that the expansion is adiabatic. This requires that both the entropy and the total number of photons remain constant. The entropy of black body radiation is

$$S = \frac{4U}{3T}, \qquad U = aVT^4, \tag{8.3}$$

where $a = 7.56 \times 10^{-16}\,\mathrm{J\,K^{-1}m^{-3}}$ is the Stefan-Boltzmann constant. With $V \propto (1+z)^{-3}$ and $T \propto 1+z$ it follows that S is independent of z, so that indeed the entropy is constant. The energy per photon is proportional to T, so that the number of photons is proportional to U/T, which is also independent of z, as required.

The entropy of the Universe is dominated by the entropy of the CMB.

8.4 Fluctuations in the CMB

Although the CMB is extremely close to a black body spectrum there are small deviations from a black body spectrum. The largest of these (the 'dipole' anisotropy) has a simple interpretation. We must be moving relative to the rest frame of the CMB due to the motion of the Earth around the Sun, the Sun around the Galaxy, the Galaxy in the local cluster of galaxies, and the local cluster relative to distant clusters. Our motion relative to the CMB can be measured due the fact that photons coming from the direction in which we are moving are blue shifted and photons coming from the opposite direction are red shifted. More generally, the observed temperature, relative to the temperature T_0 in the rest frame of the CMB, depends on angle, θ, relative to our direction of motion, and is given by

$$T_{\rm obs} = \frac{T_0}{\gamma[1-(v/c)\cos\theta]}, \tag{8.4}$$

with $\gamma = [1-(v/c)^2]^{-1/2}$.

Our velocity relative to the rest frame of the CMB has been measured (from data obtained by the COBE experiment) to be $370\,\mathrm{km\,s^{-1}}$ in a specific direction. The interpretation of this remains somewhat uncertain because other estimates of our motion suggest that it is $\sim 600\,\mathrm{km\,s^{-1}}$ in a different direction.

Measurements of very small distortions in the CMB provide other information of the structure and history of the Universe. One particular type of distortion (the Sunyaev-Zel'dovich effect) was predicted due to scattering of electrons in hot gas in large clusters of galaxies, and this effect has been observed. Another type of fluctuation is predicted: inhomogeneities in the Universe at the epoch of recombination should produce fluctuations in the CMB. The COBE experiment measured fluctuations in the CMB on scales of tens of degrees across the sky. However, COBE did not have the resolution to measure the most important fluctuations which are predicted to be on a scale $\sim 1^\circ$. Spacecraft with instruments designed to measure these fluctuations will be launched within the next year or so.

8.5 Black holes

A black hole is a solution of Einstein's equation for a point mass M. Three classes of black holes are of interest: supermassive black holes, stellar mass black holes and mini black holes. Supermassive black holes are present in the centers of so-called active galaxies: accretion of matter onto the black hole powers the extremely energetic phenomena associated with quasars and other active galaxies. The observed masses vary from a modest $2 \times 10^6\,M_\odot$ ($1\,M_\odot = 2.0 \times 10^{30}\,\mathrm{kg}$) for the central black hole in the Milky Way to $> 10^9\,M_\odot$ for some quasars. Stellar mass black holes are Galactic objects. They are detected due to their X-ray emission, they have masses $\sim 10\,M_\odot$, and they are thought to be formed in supernova explosions when the mass of the exploding star is too large for the remnant to be a neutron star. Mini black holes are objects that could have formed in the early phases of the evolution of the Universe. There is no direct evidence for the existence of mini black holes.

The essential idea of a black hole may be understood by considering the escape velocity from the Earth. At the surface of the Earth (or at any radial distance above the surface) the gravitational potential is $-GM/r$, where $G = 6.67^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$ is Newton's constant and M is the mass of the Earth. If one propels any mass m in the radial direction with a speed v it falls back to the surface of the Earth if its kinetic energy is less than its gravitational potential energy, that is, for $\frac{1}{2}mv^2 < GMm/r$ and it escapes to infinity if its kinetic energy is greater than its gravitational potential energy, that is, for $\frac{1}{2}mv^2 > GMm/r$. The speed

$$v_{\mathrm{esc}} = \left(\frac{2GM}{r}\right)^{1/2} \tag{8.5}$$

is the escape speed. This calculation implies that, for objects other than the Earth, the escape speed increases as M/r increases. For example, the mass of a typical neutron star is $1.4\,M_\odot$, so that (8.6) implies $r_g \sim 5\,\mathrm{km}$, and the radius of a neutron star, $\sim 10\,\mathrm{km}$, is only a factor of two or so larger.

The nonrelativistic calculation leading to (8.5) obviously breaks down for $2GM/r > c^2$, or for $r < r_g$, where

$$r_g = \frac{2GM}{c^2} = 3 \times 10^3 \left(\frac{M}{M_\odot}\right)\ \mathrm{m} \tag{8.6}$$

is the gravitational radius for the mass M. A black hole may be thought of as an object whose actual radius is smaller than its gravitational radius. Nothing, not even light, can escape from a black hole. However, there is no restriction on matter (or radiation) falling through the radius r_g into the black hole.

8.6 Thermodynamics of black holes

There are conceptual difficulties with radiation and black holes. One difficulty concerns the entropy when matter falls into a black hole. The entropy associated with the matter is lost to the remainder of the Universe, and this appears to be inconsistent with the requirement that the entropy of the Universe never decreases. Another difficulty is that a black hole absorbs radiation but evidently does not emit any. This is contrary to what one expects on the basis of the second law of thermodynamics applied to systems that radiate. The second law suggests that a thermal distribution of radiators at a specific temperature should be in equilibrium with black body radiation at that temperature. Then emission is balanced by absorption. However, this does not appear to be the case in the classical theory of black holes. These difficulties are overcome by assigning a temperature and an entropy to a black hole.

Stephen Hawking argued that black holes can in fact radiate ("Hawking radiation") when quantum effects are taken into account. Hawking radiation involves the creation of electron-positron pairs. Quantum electrodynamics implies that in a vacuum there are fluctuations in which virtual electrons and positrons are created and destroyed on times δt such that one has $\delta t\,\delta E \sim \hbar$, with $\delta E \sim 2mc^2$, where mc^2 is the rest energy of an electron. It is well established that these virtual pairs can become real pairs in the presence of an external electric field. Hawking argued that this must also be the case in the presence of a gravitational field. The electron and positron are created very close to each other but not exactly at the same point. For pairs created near $r = r_g$, it is possible for one particles (randomly either the electron or the positron) to be created at $r < r_g$ and the other to be created at $r > r_g$. The former is trapped and falls into the black hole, and the latter can escape. Hawking radiation consists of such escaping electrons and positrons.

Granted the existence of Hawking radiation, we must be able to assign a temperature to a black hole such that it is in equilibrium between emission (due to Hawking radiation) and absorption at this temperature. On dimensional grounds, this temperature must be such that the wavelength, $\hbar c/kT$, of thermal radiation at this temperature is of order the gravitational radius, $2GM/c^2$. The numerical value is determined by a more detailed argument, and it gives

$$T = \frac{\hbar c^3}{8\pi GMk} \approx \frac{10^{-7}}{(M/M_\odot)}\,\mathrm{K}. \tag{8.7}$$

A sound basis for the concept of the entropy of a black hole was provided by a theorem proved by Hawking: the area of a black hole never decreases. Thus, if the entropy of the black hole is proportional to its area, then the entropy of a black hole never decreases, thereby satisfying an essential requirement for the entropy. The area of a black hole is taken to be

$$A = 4\pi r_g^2 = \frac{16\pi G^2M^2}{c^4}. \tag{8.8}$$

We can determine the entropy of a black hole by noting that a change in its mass by dM implies a change in its energy by $d(Mc^2)$ and equating this change to the amount of heat. Thus we have

$$d(Mc^2) = TdS, \qquad dM = \frac{c^4}{32\pi G^2 M}\, dA, \tag{8.9}$$

where the second expression follows from (8.8). Now using (8.7) one finds

$$S = \frac{4\pi G M^2 k}{\hbar c}. \tag{8.10}$$

These arguments overcomes the formal difficulty with black holes and thermodynamic concepts.

For supermassive and stellar mass black holes the temperature (8.7) is exceedingly small and the entropy (8.10) is exceedingly large. The temperature of a black hole is so small that it cannot be in thermal equilibrium with black body radiation at any temperature of practical interest. The entropy of a black hole is very large and it increases when the black hole swallows any matter (thereby increasing M). This suggests that in practice supermassive and Galactic black holes are very far from thermodynamic equilibrium with their surroundings. However, for mini black holes the situation is more interesting.

8.6.1 Evaporation of black holes

Hawking radiation consists of electrons and positrons, and it becomes important when the Compton wavelength of the electron is of order the gravitational radius of the black hole. This corresponds to

$$\lambda_c = \frac{\hbar}{mc} \sim \frac{GM}{c^2}, \tag{8.11}$$

The power radiated in Hawking radiation can be estimated from the standard formula for the power radiated by a black body, $\propto AT^4$. With $A \propto M^2$ and $T \propto M^{-1}$, this implies a power $\propto M^{-2}$. Thus Hawking radiation increases with decreasing mass of the black hole. The emission of the radiation causes the mass to decrease, with the power equal to $-c^2 dM/dt$. Thus Hawking radiation causes a black hole to radiate away its mass, so that the mass decreases at a rate $dM/dt \propto -M^{-2}$. On integrating this expression one finds that the mass reduces to zero in a finite time, which is identified as

$$\tau \sim \frac{G^2 M^3}{\hbar c^4} \sim 10^{10} \left(\frac{M}{10^{12}\,\mathrm{kg}}\right)^3 \mathrm{yr}. \tag{8.12}$$

It follows that black holes with masses less than about 10^{12} kg evaporate due to Hawking radiation in less than the age of the Universe.

An evaporating black hole would give an intense burst of high energy radiation. There is no convincing evidence that this predicted phenomenon has been observed, implying a restriction on the properties of mini black holes that might have been created in the Big Bang.

Exercise Set 8

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